Multipole Expansion of 2D Coulomb Potential

CSCI 653 : L. Greengard & V. Rokhlin, J. Comp. Phys. 73, 325 ('87)

Lemma 1

For $|z| > |z_0|$ where $z, z_0 \in \mathbb{C}$

$$\log(z - z_0) = \log(z) - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{z_0}{z}\right)^k \tag{1}$$

We rewrite Eq. (1) as

$$\log\Big(1 - \underbrace{\frac{z_0}{z}}_{w}\Big) = -\sum_{k=1}^{\infty} \frac{1}{k}\underbrace{\left(\frac{z_0}{z}\right)}_{w}^{k} \qquad (|w| < 1)$$

$$f(w) = \log(1-w) \xrightarrow[w \to 0]{} 0$$

$$f'(w) = -\frac{1}{1-w} \xrightarrow[w \to 0]{} -1$$

$$f^{(2)}(w) = -(1-w)^{-2} \xrightarrow[w \to 0]{} -1$$

$$f^{(3)}(w) = -2(1-w)^{-3} \rightarrow -2$$

$$f^{(4)}(w) = -2\cdot 3(1-w)^{-4} \rightarrow -2\cdot 3$$

$$\vdots$$

$$f^{(k)}(w) = -(k-1)!\,(1-w)^{-k} \rightarrow -(k-1)!$$

$$\therefore f(w) = \underbrace{\cancel{f(0)}}_{0} + \sum_{k=1}^{\infty} \frac{f^{(k)}(w)}{k!} w^k = -\sum_{k=1}^{\infty} \frac{1}{k} w^k \qquad (|w| < 1) \quad //$$

Lemma 2 (Multipole Expansion)

Suppose m charges of strengths $\{q_i, i = 1, \dots, m\}$ are located at points $\{z_i, i = 1, \dots, m\}$, with $|z_i| < r$. Then, for any $z \in \mathbb{C}$ with $|z| > r$, the potential is given by

$$\phi(z) = Q \log(z) + \sum_{k=1}^{\infty} \frac{a_k}{z^k} \tag{2}$$

where

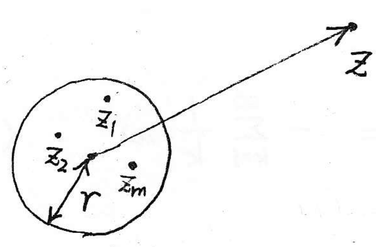
$$Q = \sum_{i=1}^{m} q_i \qquad \text{and} \qquad a_k = \sum_{i=1}^{m} \frac{-q_i z_i^k}{k} \tag{3}$$

Furthermore for any $p \geq 1$,

$$\left| \phi(z) - Q \log(z) - \sum_{k=1}^{p} \frac{a_k}{z^k} \right| \leq \alpha \left| \frac{r}{z} \right|^{p+1} \leq \left(\frac{A}{c-1} \right) \left(\frac{1}{c} \right)^{p} \tag{4}$$

where

$$c = \left| \frac{z}{r} \right|, \quad A = \sum_{i=1}^{m} |q_i|, \quad \text{and} \quad \alpha = \frac{A}{1 - |r/z|} \tag{5}$$



① $\phi(z) = \sum_{i=1}^{m} q_i \underbrace{\log(z-z_i)}$

$$\log(z) - \sum_{k=1}^{\infty} \frac{1}{k} \frac{z_i^k}{z^k}$$

$$= \underbrace{\left(\sum_{i=1}^{m} q_i\right)}_{Q} \log(z) + \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{z^k} \underbrace{\left(-\sum_{i=1}^{m} q_i z_i^k\right)}_{a_k}$$

② $\left| \phi(z) - Q\log(z) - \sum_{k=1}^{p} \frac{a_k}{z^k} \right|$

$$= \left| \sum_{k=p+1}^{\infty} \frac{a_k}{z^k} \right|$$

$$\leq \sum_{k=p+1}^{\infty} \left| \frac{a_k}{z^k} \right| \leq \sum_{k=p+1}^{\infty} \frac{1}{|z|^k} \underbrace{\left| -\sum_{i=1}^{m} q_i z_i^k \right|}_{\leq \underbrace{\sum_{i=1}^{m} |q_i|}_{A} r^k} \leq A \sum_{k=p+1}^{\infty} \left| \frac{r}{z} \right|^k$$

$$= A \frac{\left|\frac{r}{z}\right|^{p+1}}{1 - \left|\frac{r}{z}\right|} = \frac{A/c}{1-1/c} \left(\frac{1}{c}\right)^p \quad //$$

Lemma 3 (Shifting the center of a multipole expansion)

Suppose that

$$\phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k} \qquad (6)$$

is a multipole expansion of the potential due to a set of m charges $\{q_i, i = 1, \ldots, m\}$, all of which are located inside the circle D of radius R with center at z_0.

Then for z outside the circle D_1 of radius $(R + |z_0|)$ and center at the origin,

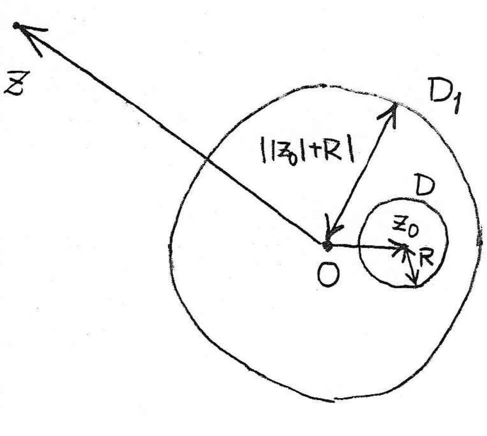
$$\phi(z) = a_0 \log(z) + \sum_{l=1}^{\infty} \frac{b_l}{z^l} \qquad (7)$$

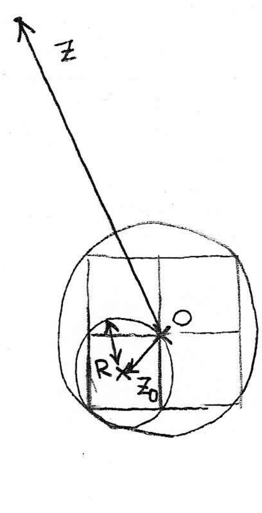
where

$$b_l = \sum_{k=1}^{l} {}_{l-1}C_{k-1}\, a_k\, z_0^{l-k} - \frac{a_0 z_0^l}{l} \qquad (8)$$

Furthermore, for any $p \geq 1$,

$$\left| \phi(z) - a_0 \log(z) - \sum_{l=1}^{p} \frac{b_l}{z^l} \right| \leq \frac{A}{1 - \left| \frac{|z_0| + R}{z} \right|} \left| \frac{|z_0| + R}{z} \right|^{p+1} \qquad (9)$$





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① $$\phi(z) = \underbrace{a_0 \log(z-z_0)} + \sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k}$$

$$a_0 \log(z) - \sum_{l=1}^{\infty} \frac{a_0}{l}\left(\frac{z_0}{z}\right)^l + \sum_{k=1}^{\infty} a_k \underbrace{\frac{1}{(z-z_0)^k}}$$

$$f(z_0) = \frac{1}{(z-z_0)^k} \xrightarrow[z_0=0]{} \frac{1}{z^k}$$

$$f'(z_0) = k(z-z_0)^{-k-1} \rightarrow \frac{k}{z^{k+1}}$$

$$f''(z_0) = (k+1)k(z-z_0)^{-k-2} \rightarrow \frac{(k+1)k}{z^{k+2}}$$

$$\vdots$$

$$f^{(l)}(z_0) = \frac{(k+l-1)!}{(k-1)!}(z-z_0)^{-k-l}$$

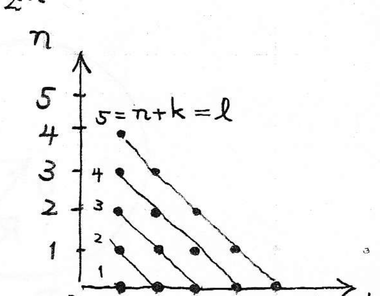
$$\rightarrow \frac{(k+l-1)!}{(k-1)!}\frac{1}{z^{k+l}}$$

$$= a_0\log(z) - \sum_{l=1}^{\infty}\frac{a_0}{l}\left(\frac{z_0}{z}\right)^l + \sum_{k=1}^{\infty} a_k \underbrace{\left[\frac{1}{z^k} + \sum_{l=1}^{\infty}\frac{(l+k-1)!}{l!(k-1)!}\frac{1}{z^{k+l}} z_0^l\right]}_{\sum_{l=0}^{\infty} {}_{l+k-1}C_{k-1} \frac{1}{z^{k+l}} z_0^l}$$

$$= a_0\log(z) - \sum_{l=1}^{\infty}\frac{a_0}{l}\left(\frac{z_0}{z}\right)^l + \underbrace{\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} a_k\, {}_{n+k-1}C_{k-1}\frac{z_0^n}{z^{k+n}}}_{l \leftarrow n+k}$$

$$\sum_{l=1}^{\infty}\sum_{k=1}^{l} a_k\, {}_{l-1}C_{k-1}\frac{z_0^{l-k}}{z^l}$$

$$= a_0\log(z) + \sum_{l=1}^{\infty}\Big(\underbrace{\sum_{k=1}^{l} a_k\, {}_{l-1}C_{k-1} z_0^{l-k} - \frac{a_0}{l} z_0^l}_{b_l}\Big)\frac{1}{z^l}$$



② Ordinary multipole expansion around D_1 //

Lemma 4 (Local Taylor Expansion of Multipole Potential)

Suppose that m charges of strengths $\{q_i, i=1,\ldots,m\}$ are located inside the circle D_1 with radius R_1 and center at z_0, and $|z_0| > (c+1)R$ with $c > 1$. Then the corresponding multipole expansion (6) converges inside the circle D_2 of radius R centered about the origin. Inside D_2, the potential due to the charge is described by a power series:

$$\phi(z) = \sum_{l=0}^{\infty} b_l z^l \tag{10}$$

where

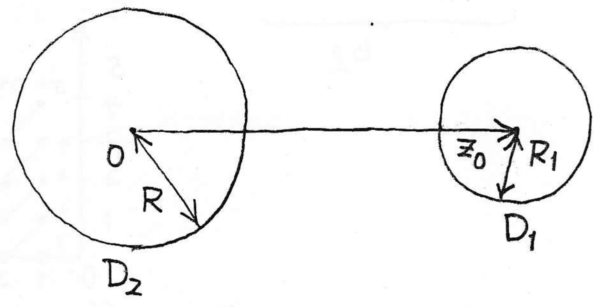
$$b_0 = \sum_{k=1}^{\infty} \frac{a_k}{z_0^k}(-1)^k + a_0 \log(-z_0) \tag{11}$$

and

$$b_l = \left[\frac{1}{z_0^l}\sum_{k=1}^{\infty}\frac{a_k}{z_0^k}\, {}_{l+k-1}C_{k-1}\,(-1)^k\right] - \frac{a_0}{l\cdot z_0^l} \quad (l \geq 1) \tag{12}$$

Furthermore, for any $p \geq \max\left(2, \frac{2c}{c-1}\right)$,

$$\left|\phi(z) - \sum_{l=0}^{p} b_l \cdot z^l\right| < \frac{A\,[4e(p+c)(c+1)+c^2]}{c\,(c-1)}\left(\frac{1}{c}\right)^{p+1} \tag{13}$$



$$\phi(z) = a_0 \underbrace{\log(z-z_0)} + \sum_{k=1}^{\infty} \underbrace{\frac{a_k}{(z-z_0)^k}}$$

$$f(z) = \log(z-z_0) \xrightarrow[z=0]{} \log(-z_0)$$

$$f'(z) = \frac{1}{z-z_0} \rightarrow -\frac{1}{z_0}$$

$$f''(z) = -(z-z_0)^{-2} \rightarrow -\frac{1}{z_0^2}$$

$$f^{(3)}(z) = 2(z-z_0)^{-3} \rightarrow -\frac{2}{z_0^3}$$

$$\vdots$$

$$f^{(\ell)}(z) = (-1)^{\ell-1}(\ell-1)!\,(z-z_0)^{-\ell} \rightarrow -\frac{(\ell-1)!}{z_0^\ell}$$

$$f(z) = (z-z_0)^{-k} \xrightarrow[z=0]{} (-1)^k \frac{1}{z_0^k}$$

$$f'(z) = -k(z-z_0)^{-k-1} \rightarrow (-1)^k \frac{k}{z_0^{k+1}}$$

$$f''(z) = k(k+1)(z-z_0)^{-k-2} \rightarrow (-1)^k \frac{k(k+1)}{z_0^{k+2}}$$

$$\vdots$$

$$f^{(\ell)}(z) = \frac{(\ell+k-1)!}{(k-1)!}(z-z_0)^{-k-\ell} \rightarrow (-1)^k \frac{(\ell+k-1)!}{(k-1)!} \frac{1}{z_0^{\ell+k}}$$

$$\phi(z) = a_0 \log(-z_0) - \sum_{\ell=1}^{\infty} \frac{a_0}{\ell z_0^\ell} z^\ell + \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty} a_k (-1)^k \underbrace{\frac{(\ell+k-1)!}{\ell!\,(k-1)!}}_{{}_{\ell+k-1}C_{k-1}} \frac{1}{z_0^{\ell+k}} z^\ell$$

$$= a_0 \log(-z_0) + \sum_{k=1}^{\infty} a_k (-1)^k \frac{1}{z_0^k}$$

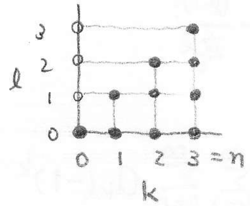
$$+ \sum_{\ell=1}^{\infty} \left[\frac{-a_0}{\ell z_0^\ell} + \sum_{k=1}^{\infty} a_k (-1)^k {}_{\ell+k-1}C_{k-1} \frac{1}{z_0^{\ell+k}} \right] \quad //$$

Lemma 5 (Shifting the origin of Taylor expansion)

For any $z, z_0 \in \mathbb{C}$, and $\{a_k, k=1,\dots,n\}$,

$$\sum_{k=0}^{n} a_k (z-z_0)^k = \sum_{l=0}^{n} \left[\sum_{k=l}^{n} a_k \, {}_kC_l \, (-z_0)^{k-l} \right] z^l \qquad (14)$$

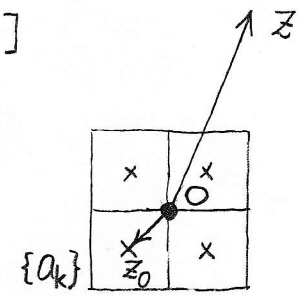
$$\because \quad \sum_{k=0}^{n} a_k \underbrace{(z-z_0)^k}_{\sum_{l=0}^{k} {}_kC_l \, z^l (-z_0)^{k-l}} \qquad //$$



$$= \sum_{l=0}^{n} z^l \sum_{k=0}^{n} a_k \, {}_kC_l \, (-z_0)^{k-l}$$

(Example)

[I]



$$\Phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k}$$

shifting the origin of a multipole expansion

$$\begin{cases} a_0 = \sum_{i=1}^{m} q_i \\ a_k = \sum_{i=1}^{m} \frac{-q_i (z_i - z_0)^k}{k} \end{cases}$$

$$\Phi(z) = a_0 \log(z) + \sum_{l=1}^{\infty} \frac{b_l}{z^l}$$

$$b_l = \left[\sum_{k=1}^{l} a_k \, z_0^{l-k} \, {}_{l-1}C_{k-1} \right] - \frac{a_0 z_0^l}{l}$$

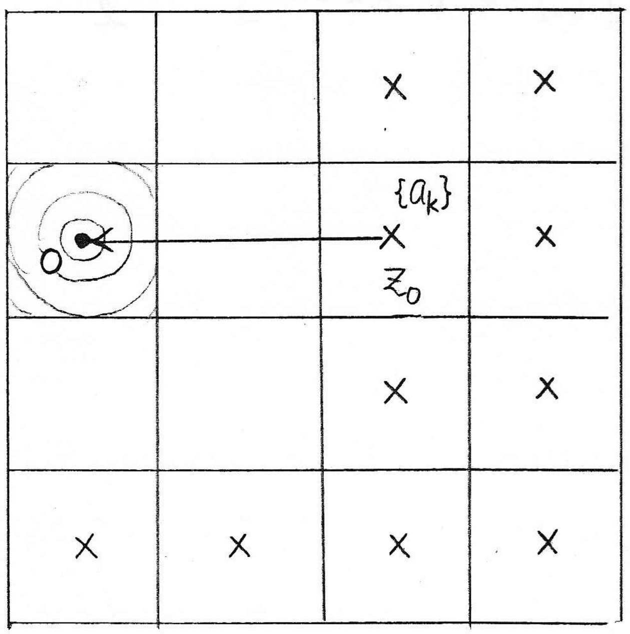
[II]

$$\phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k}$$

$$\Big\downarrow \text{ local Taylor expansion}$$

$$\phi(z) = \sum_{l=0}^{\infty} b_l z^l$$

$$\begin{cases} b_0 = \displaystyle\sum_{k=1}^{\infty} \frac{a_k}{z_0^k} (-1)^k + a_0 \log(-z_0) \\ b_l = \left[\displaystyle\frac{1}{z_0^l} \sum_{k=1}^{\infty} \frac{a_k}{z_0^k} \, {}_{l+k-1}C_{k-1} (-1)^k \right] - \frac{a_0}{l z_0^l} \quad (l \geq 1) \end{cases}$$



[III]

$$\phi(z) = \sum_{k=0}^{n} a_k (z - z_0)^k$$

$$\Big\downarrow \text{ shifting the origin of Taylor expansion}$$

$$\phi(z) = \sum_{l=0}^{n} b_l z^l$$

$$b_l = \sum_{k=l}^{n} a_k \, {}_kC_l (-z_0)^{k-l}$$