Electrostatic Potential around a Charged Line

- Let

$$\rho(x,y,\cancel{z}) = q\,\delta(x)\,\delta(y) \tag{1}$$

be the charge distribution, where $\delta(x)$ is the Dirac's delta function,

$$\int_{-\infty}^{\infty} dx\,\delta(x)\,{}^{\forall}f(x) = f(0) \tag{2}$$

and q has the dimension of charge / length. → line charge density.

- The electrostatic potential $\phi(x,y,\cancel{z})$ (it won't depend on z) is determined from

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \cancel{\frac{\partial^2}{\partial z^2}}\right)\phi(x,y,\cancel{z}) = -4\pi\rho(x,y,\cancel{z}) \tag{3}$$

Let the 2D gradient vector be

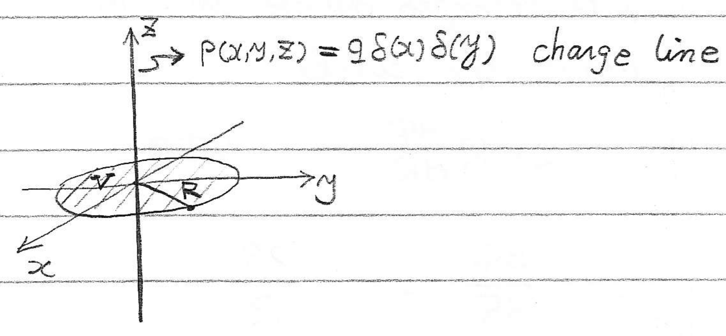
$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{4}$$

and Laplacian

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5}$$

Then, Eq. (3) can be rewritten as

$$\nabla^2\phi(x,y) = -4\pi q\,\delta(x)\,\delta(y) \tag{6}$$

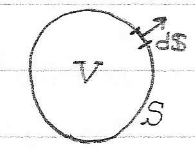


— Let's integrate Eq.(6) in the circular area with radius R.

$$\int_V d^2r\, \nabla^2\phi(x,y) = -4\pi q \underbrace{\int_V d^2r\, \delta(x)\,\delta(y)}_{=1} = -4\pi q \tag{7}$$

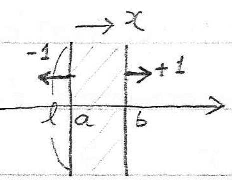
Now use Gauss' theorem. $\left(\int_V d^2r\, \nabla = \int_S d\mathbb{S} \right)$ $d\mathbb{S}$ → surface element

$$\int_V d^2r\, \nabla\cdot\nabla\phi(x,y) = \int_S d\mathbb{S}\cdot\nabla\phi(x,y) \tag{8}$$



where $d\mathbb{S}$ is the surface areal element vector normal to the surface. Note if the surfaces are normal to the x axis,

$$\int_V d^2r\, \underbrace{\nabla}_{\frac{\partial}{\partial x}+\frac{\partial}{\partial \cancel{y}}} f(x,\cancel{y})$$



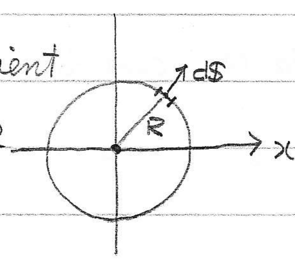
$$= \left(l \int_a^b dx \frac{df}{dx},\ 0 \right)$$ ← now only look at x-component of the vector

$$= \left(l\, [f(x)]_a^b, 0\right) = \left(l\,[f(b) - f(a)], 0\right) = \left(\underbrace{l f(b) + (-l) f(a)}_{\text{integration of areal element}}, 0\right) = \int_S d\mathbb{S}\, f$$

— Substituting Eq.(8) in (7)

$$\int_S d\mathbb{S}\cdot\nabla\phi(x,y) = -4\pi q \tag{9}$$

From the symmetry, the gradient vector is normal to the surface with strength $d\phi/dR$ uniform across the circle



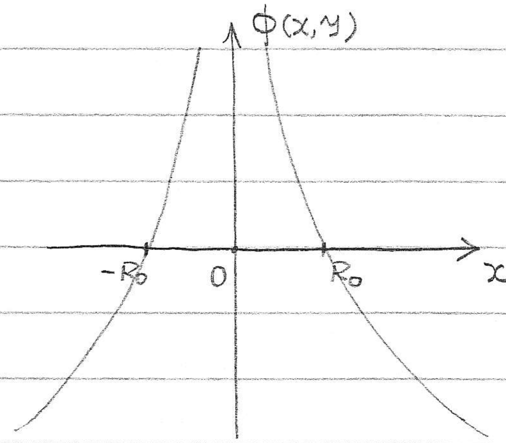
$$\therefore\ \cancel{2\pi} R \frac{d\phi}{dR} = -\overset{2}{\cancel{4\pi}} q \tag{10}$$

$$\therefore\ \frac{d\phi}{dR} = -\frac{2q}{R} \tag{11}$$

$$\therefore \phi(R) = -2q\log R + C \tag{12}$$

where C is the integration constant. Setting $C = 2q\log R_0$,

$$\phi(R) = -2q\log\left(\frac{R}{R_0}\right) \tag{13}$$



— Measure the length in unit of R_0 and $+2q \to q$ (how we measure the charge density or define the 2D problem)

$$\phi(x,y) = -q\log\sqrt{x^2+y^2} \tag{14}$$

Let $z = x+iy = re^{i\theta}$ where $r=\sqrt{x^2+y^2}$ $\tan\theta = y/x$ then

$$\phi(x,y) = -\mathrm{Re}\,\underbrace{q\log z} \tag{15}$$

$$\therefore \log(re^{i\theta}) = \log r + i\theta.$$

This is the use of complex $\log z$ to compute 2D electrostatic potential.