Minimal Complex Analysis

- **Complex function:** A mapping from a complex variable $z = x + iy$ ($i = \sqrt{-1}$) to a complex number $f(z) \in \mathbf{C}$.
- **Differentiation:** A complex function $f(z)$ at z is *differentiable* if the quantity

$$\frac{f(z+\delta z) - f(z)}{\delta z}$$

converges to a unique value as $\delta z \to 0$, independent of how $\delta z = \delta x + i\delta y$ approaches 0 (i.e., independent of the ratio $\delta x/\delta y$). Then the *derivative* of $f(z)$ is defined as

$$\frac{df}{dz} = f'(z) \equiv \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}. \tag{1}$$

- **Cauchy-Riemann conditions**: Let

$$f(z) = \mathrm{Re}\, f(z) + i\,\mathrm{Im}\, f(z) = f_1(x,y) + i f_2(x,y), \tag{2}$$

then $f(z)$ is differentiable at z if the following conditions are satisfied:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}. \tag{3}$$

$\because$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\frac{\partial f_1}{\partial x}\delta x + \frac{\partial f_1}{\partial y}\delta y + i\left(\frac{\partial f_2}{\partial x}\delta x + \frac{\partial f_2}{\partial y}\delta y\right)}{\delta x + i\delta y}$$

$$= \frac{\left(\frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}\right) + \left(\frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}}$$

By substituting Eq. (3) into the above equation in such a way that all y derivatives are eliminated, we obtain

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\left(\frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}\right) + \left(-\frac{\partial f_2}{\partial x} + i\frac{\partial f_1}{\partial x}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}} = \frac{\frac{\partial f_1}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right) + i\frac{\partial f_2}{\partial x}\left(1 + i\frac{\delta y}{\delta x}\right)}{1 + i\frac{\delta y}{\delta x}} = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x},$$

which is independent of $\delta x/\delta y$.//

(Corollary) When the derivative exists, the above proof shows that it is given by

$$\begin{aligned} f'(z) &= f_1'(x,y) + i f_2'(x,y) \\ &= \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x} \\ &= \frac{\partial f_1}{\partial x} - i\frac{\partial f_1}{\partial y} \end{aligned}. \tag{4}$$

The last equality is a consequence of the Cauchy-Riemann condition, Eq. (3), and is useful when complex analysis is used as a means to obtain a gradient of a 2-dimmensional real function,

$$\nabla f_1(x, y) = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}\right) \tag{5}$$

(Theorem) The complex power function $f(z) = z^n$ ($n = 1, 2, ...$) is differentiable.

∵ Proof by mathematical induction.

(I) $n = 1$

Let $f(z) = z = x + iy = f_1 + if_2$. Then,

$$\frac{\partial f_1}{\partial x} = 1 = \frac{\partial f_2}{\partial y}, \frac{\partial f_1}{\partial y} = 0 = -\frac{\partial f_2}{\partial x}$$

$\therefore z^1$ is differentiable.

(II) Assume that $g(x) = z^k$ is differentiable, and consider $f(z) = z^{k+1}$.

$$f(z) = zg(z) = (x + iy)(g_1 + ig_2) = (xg_1 - yg_2) + i(xg_2 + yg_1) = f_1 + if_2.$$

$$\begin{aligned}\frac{\partial f_2}{\partial y} &= x\frac{\partial g_2}{\partial y} + g_1 + y\frac{\partial g_1}{\partial y} \\ &= x\frac{\partial g_1}{\partial x} + g_1 - y\frac{\partial g_2}{\partial x} \quad \left(\because \frac{\partial g_2}{\partial y} = \frac{\partial g_1}{\partial x}; \frac{\partial g_1}{\partial y} = -\frac{\partial g_2}{\partial x}\right) \\ &= \frac{\partial f_1}{\partial x}\end{aligned}$$

$$\begin{aligned}\frac{\partial f_2}{\partial x} &= g_2 + x\frac{\partial g_2}{\partial x} + y\frac{\partial g_1}{\partial x} \\ &= g_2 - x\frac{\partial g_1}{\partial y} + y\frac{\partial g_2}{\partial y} \quad \left(\because \frac{\partial g_2}{\partial x} = -\frac{\partial g_1}{\partial y}; \frac{\partial g_1}{\partial x} = \frac{\partial g_2}{\partial y}\right) \\ &= -\frac{\partial f_1}{\partial y}\end{aligned}$$

$\therefore$ If z^k is differentiable, then z^{k+1} is differentiable.

(III) Recursive application of (II), starting with (I), proves that z^n is differentiable for $\forall n \geq 1$.//

(Theorem) $\frac{d}{dz} z^n = nz^{n-1}$. (6)

∵ Proof by mathematical induction.

(I) $n = 1$

$$\frac{\delta z}{\delta z} = \frac{\delta x + i\delta y}{\delta x + i\delta y} = 1.$$

(II) Assume that Eq. (6) is true for $n = k$, then

$$\frac{d}{dz} z^{k+1} = \frac{d}{dz}\left(z \bullet z^k\right) = z^k + z \bullet kz^{k-1} = (k+1)z^k.$$

$\therefore$ If Eq. (6) is true for $n = k$, then it is also true for $n = k+1$.

(III) Recursive application of (II), starting with (I), proves that Eq. (6) is true for $\forall n \geq 1$.//

(Definition) $e^z \equiv \sum_{n=0}^{\infty} \frac{z^n}{n!}$ (7)

(Theorem) e^z is differentiable and $\frac{d}{dz}e^z = e^z$. (8)

$\because$

$$\frac{d}{dz}\left(1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots\right) = 1 + z + \frac{1}{2!}z^2 + \cdots = e^z.//$$

(Definition) $w = \log z$ is defined as the inverse of $z = e^w$.

(Theorem) $\log z$ is differentiable and $\frac{d}{dz}\log z = \frac{1}{z}$. (9)

$\because$

$$\delta z = e^w \delta w$$

$$\therefore \frac{\delta w}{\delta z} = \frac{1}{e^w} = \frac{1}{z}.//$$

(Theorem) $\log(1 - z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$. (10)

$\because$

$$f(z) = \log(z)$$

$$f^{(1)}(z) = \frac{df}{dz} = 1/z$$

$$f^{(2)}(z) = \frac{d^2 f}{dz^2} = -1/z^2$$

$$f^{(3)}(z) = 2/z^3$$

$$f^{(4)}(z) = -3!/z^4$$

$$\vdots$$

$$f^{(n)}(z) = -(-1)^n (n-1)!/z^n$$

$$\therefore f(1 - z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(-z)^k = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \cdots = -\sum_{k=1}^{\infty} \frac{z^k}{k}.//$$
