Quantum Fourier Transform

Consider n-qubit quantum states $|x\rangle$, where $x \in \{0,1,\dots,N-1\}$ ($N = 2^n$) is an integer corresponding to binary representation of the n-qubit states:

$$x = x_{n-1}x_{n-2}\cdots x_1x_0 = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0 \ \ (x_0, x_1, \cdots, x_{n-1} \in \{0,1\}). \qquad (1)$$

Quantum Fourier transform (QFT) acting on the basis states is defined as

$$\text{QFT: } |x\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi kx}{N}} |k\rangle. \qquad (2)$$

On a more general state defined through a function $f(x)$, QFT then acts as

$$\text{QFT: } \sum_{x=0}^{N-1} f(x)|x\rangle \to \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{i\frac{2\pi kx}{N}} f(x) \right) |k\rangle = \sum_{k=0}^{N-1} \tilde{f}(k)|k\rangle \qquad (3)$$

where

$$\tilde{f}(k) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{i\frac{2\pi kx}{N}} f(x) \qquad (4)$$

is an ordinary discrete Fourier transform (DFT).

Now consider the product, $kx/N = kx/2^n$, in Eq. (2):

$$\frac{kx}{2^n} = \frac{(k_{n-1}2^{n-1} + k_{n-2}2^{n-2} + \cdots + k_1 2^1 + k_0 2^0)(x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0)}{2^n}$$
$$= (k_{n-1}2^{n-1} + k_{n-2}2^{n-2} + \cdots + k_1 2^1 + k_0 2^0) \left(\frac{x_{n-1}}{2} + \frac{x_{n-2}}{2^2} + \cdots + \frac{x_1}{2^{n-1}} + \frac{x_0}{2^n} \right). \qquad (5)$$

Noting that the integer part of kx/N does not contribute to $e^{i\frac{2\pi kx}{N}}$, Eq. (5) can be simplified as

$$\frac{kx}{2^n} = k_{n-1}(.x_0) + k_{n-2}(.x_1x_0) + \cdots + k_1(.x_{n-2}x_{n-3}\cdots x_0) + k_0(.x_{n-1}x_{n-2}\cdots x_0), \qquad (6)$$

where the factors in parentheses are binary representation of fraction:

$$.x_m x_{m-1} \cdots x_0 = \frac{x_m}{2} + \frac{x_{m-1}}{2^2} + \cdots + \frac{x_0}{2^{m+1}}. \qquad (7)$$

Substituting Eq. (6) in Eq. (2), we obtain

$$\text{QFT: } |x\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i\frac{2\pi kx}{N}} |k\rangle = \frac{1}{\sqrt{2^n}} \sum_{k_{n-1}=0}^{1} \cdots \sum_{k_0=0}^{1} e^{i2\pi\left(k_{n-1}(.x_0) + \cdots + k_0(.x_{n-1}x_{n-2}\cdots x_0)\right)} |k_0\rangle$$
$$= \frac{1}{\sqrt{2^n}} \sum_{k_{n-1}=0}^{1} e^{i2\pi k_{n-1}(.x_0)} |k_{n-1}\rangle \otimes \sum_{k_{n-2=0}}^{1} e^{i2\pi k_{n-2}(.x_1x_0)} |k_{n-2}\rangle \otimes \cdots \otimes \sum_{k_0=0}^{1} e^{i2\pi k_{n-1}(.x_0)} |k_0\rangle$$
$$= \frac{1}{\sqrt{2^n}} \Bigg(\underbrace{|0\rangle + e^{i2\pi(.x_0)}|1\rangle}_{k_{n-1}} \Bigg) \otimes \Bigg(\underbrace{|0\rangle + e^{i2\pi(.x_1x_0)}|1\rangle}_{k_{n-2}} \Bigg) \otimes \cdots \otimes \Bigg(\underbrace{|0\rangle + e^{i2\pi(.x_{n-1}x_{n-2}x_{n-3}\cdots x_0)}|1\rangle}_{k_0} \Bigg) . \qquad (8)$$

Equation (8) can be implemented as a quantum circuit as illustrated in Fig. 1 for 3 qubits. For j-th qubit $|x_j\rangle$, we first apply one-qubit Hadamard (H) gate as

$$\text{H: } |x_j\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_j}|1\rangle) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi(.x_j)}|1\rangle\right). \qquad (9)$$

We then apply two-qubit controlled R_1 gate where the control is $|x_{j-1}\rangle$, where

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(i\frac{\pi}{2^k}\right) \end{bmatrix}. \quad (10)$$

This transform j-th qubit into

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi(.x_j x_{j-1})}|1\rangle\right). \quad (9)$$

We continue applying controlled R_2,... with successively lower qubit until reaching x_0.

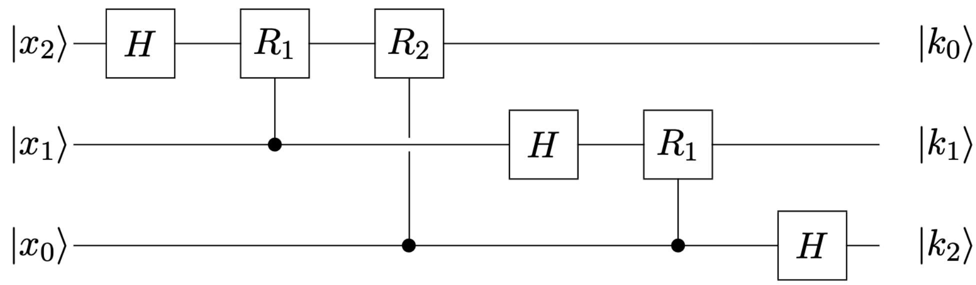

Fig. 1: Quantum Fourier transform circuit for n (= 3) qubits.

For a n-qubit state, the quantum circuit is composed of $n + (n-1) + \cdots + 2 + 1 = \frac{n(n+1)}{2} = O(n^2)$ gates. Compare this with the $O(N\log_2 N) = O(2^n n)$ arithmetic operations in the classical fast Fourier transform (FFT) algorithm (*cf.* https://aiichironakano.github.io/phys516/03QD.pdf). The QFT algorithm thus achieves an exponential reduction of computation: $2^n n/n^2 = 2^n/n$.

Reference

1. J. Preskill, *arXiv*:2106.10522 (2021).