Recursive Formula for Legendre Polynomials

Generating function

$$g(t,x) = \frac{1}{\sqrt{1-2xt+t^2}} \equiv \sum_{j=0}^{\infty} P_j(x)t^j \tag{1}$$

Recursive relation for $P_j(x)$

$$(j+1)P_{j+1} = (2j+1)xP_j - jP_{j-1} \quad P_0 = 1 \quad P_1 = x \tag{2}$$

(Proof) Differentiate Eq. (1) with respect to t.

$$\frac{\partial g}{\partial t} = \frac{2(t-x)}{-2(1-2xt+t^2)^{3/2}} = \sum_{j=1}^{\infty} jP_j(x)t^{j-1}$$

$$\therefore (x-t)\sum_{j=0}^{\infty} P_j(x)t^j = (1-2xt+t^2)\sum_{j=1}^{\infty} jP_j(x)t^{j-1}$$

$$x\sum_{j=0}^{\infty} P_j(x)t^j - \sum_{j=1}^{\infty} P_{j-1}(x)t^j = \sum_{j=0}^{\infty}(j+1)P_{j+1}(x)t^j - 2x\sum_{j=1}^{\infty} jP_j(x)t^j + \sum_{j=2}^{\infty}(j-1)P_{j-1}(x)t^j$$

Compare the coefficients of t^j:

($j = 0$)

$$xP_0 = P_1$$
$$\therefore P_0 = 1 \quad \Rightarrow \quad P_1 = x$$

($j = 1$)

$$xP_1 - P_0 = 2P_2 - 2xP_1$$
$$\therefore x^2 - 1 = 2P_2 - 2x^2 \Rightarrow P_2 = (3x^2-1)/2$$

($j \geq 2$)

$$xP_j - P_{j-1} = (j+1)P_{j+1} - 2xjP_j + (j-1)P_{j-1}$$
$$\therefore (j+1)P_{j+1} = (2j+1)xP_j - jP_{j-1}$$

Recursive relation for $P'_j(x)$

$$(x^2-1)P'_j = jxP_j - jP_{j-1} \tag{3}$$

(Proof) Differentiate Eq. (1) with respect to x.

$$\frac{\partial g}{\partial x} = \frac{-2t}{-2(1-2xt+t^2)^{3/2}} = \sum_{j=0}^{\infty} P'_j(x)t^j$$

$$\therefore \frac{t}{(1-2xt+t^2)} g(t,x) = \sum_{j=0}^{\infty} P'_j(x)t^j$$

$$\therefore t\sum_{j=0}^{\infty} P_j(x)t^j = (1-2xt+t^2)\sum_{j=0}^{\infty} P'_j(x)t^j$$

$$\sum_{j=1}^{\infty} P_{j-1}(x)t^j = \sum_{j=0}^{\infty} P_j'(x)t^j - 2x\sum_{j=1}^{\infty} P_{j-1}'(x)t^j + \sum_{j=2}^{\infty} P_{j-2}'(x)t^j$$

$$\therefore P_{j-1} = P_j' - 2xP_{j-1}' + P_{j-2}' \tag{4}$$

From the 3-point recursion, Eq. (2), for P_j (shifted by one),

$$jP_j = (2j-1)xP_{j-1} - (j-1)P_{j-2}$$

$\frac{d}{dx}\times$

$$jP_j' = (2j-1)P_{j-1} + (2j-1)xP_{j-1}' - (j-1)P_{j-2}' \tag{5}$$

$(5) + (j-1)\times(4)$ to eliminate P'_{j-2}

$$\begin{array}{rl} & jP_j' = (2j-1)P_{j-1} + (2j-1)xP_{j-1}' - (j-1)P_{j-2}' \\ +) & (j-1)P_{j-1} = (j-1)P_j' - 2(j-1)xP_{j-1}' + (j-1)P_{j-2}' \\ \hline & P_j' = jP_{j-1} + xP_{j-1}' \end{array} \tag{6}$$

Now consider another recursive formula for $P'_j(x)$ as follows:

$$\frac{\partial g}{\partial x} = \frac{-2t}{-2(1-2xt+t^2)^{3/2}} = \sum_{j=0}^{\infty} P_j'(x)t^j$$

$$\therefore t\frac{\partial g}{\partial t} = (x-t)\frac{\partial g}{\partial x}$$

$$t\sum_{j=1}^{\infty} jP_j(x)t^{j-1} = (x-t)\sum_{j=0}^{\infty} P_j'(x)t^j$$

$$\sum_{j=1}^{\infty} jP_j(x)t^j = x\sum_{j=0}^{\infty} P_j'(x)t^j - \sum_{j=1}^{\infty} P_{j-1}'(x)t^j$$

$(j = 0)$

$$0 = P_0'(x) \quad \text{OK}$$

$(j \geq 1)$

$$jP_j = xP_j' - P_{j-1}' \tag{7}$$

$(6) + x\times(7)$ to eliminate xP'_{j-1}

$$\begin{array}{rl} & P_j' = jP_{j-1} + xP_{j-1}' \\ +) & jxP_j = x^2P_j' - xP_{j-1}' \\ \hline & jxP_j - jP_{j-1} = (x^2-1)P_j' \end{array} \quad //$$