Taylor's Expansion

We use *recursive telescoping* to derive Taylor's expansion. Assume that function $f(x)$ is continuous up to the n-th derivative $f^{(n)}(x)$ in the range $[a, x]$. Then

$$\int_a^x f^{(n)}(x)dx = f^{(n-1)}(x)\Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a). \tag{1}$$

Note this is telescoping: Let $\Delta = (x-a)/N$ for a large integer N, then

$$\int_a^x f^{(n)}(x)dx \approx \sum_{i=0}^{N-1} \frac{f^{(n-1)}(a+(i+1)\Delta) - f^{(n-1)}(a+i\Delta)}{\Delta}\Delta$$

$$= f^{(n-1)}(a+\Delta) - f^{(n-1)}(a) + f^{(n-1)}(a+2\Delta) - f^{(n-1)}(a+\Delta) + \cdots + f^{(n-1)}(x) - f^{(n-1)}(x-\Delta)$$

$$= f^{(n-1)}(x) - f^{(n-1)}(a).$$

With the use of telescoping again, $\int_a^x dx \times$ Eq. (1) yields

$$\begin{aligned}\int_a^x dx \int_a^x dx f^{(n)}(x) &= \int_a^x f^{(n-1)}(x)dx - (x-a)f^{(n-1)}(a) \\ &= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a)\end{aligned}. \tag{2}$$

$\int_a^x dx \times$ Eq. (2) then yields

$$\int_a^x dx \int_a^x dx \int_a^x dx f^{(n)}(x) = f^{(n-3)}(x) - f^{(n-3)}(a) - (x-a)f^{(n-2)}(a) - \frac{(x-a)^2}{2}f^{(n-1)}(a). \tag{3}$$

Here, we have used

$$\int_a^x \frac{(x-a)^{n-1}}{(n-1)!}dx = \frac{(x-a)^n}{n!}\Big|_a^x = \frac{(x-a)^n}{n!}. \tag{4}$$

By repeating the same procedure, we eventually reach

$$\begin{aligned}\underbrace{\int_a^x dx \cdots \int_a^x dx}_{n} f^{(n)}(x) &= f(x) - f(a) - (x-a)f^{(1)}(a) - \frac{(x-a)^2}{2}f^{(2)}(a) \\ &\quad - \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)\end{aligned}.$$

By rearranging the terms, we finally obtain Taylor's expansion:

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!}f^{(k)}(a) + \underbrace{\int_a^x dx \cdots \int_a^x dx}_{n} f^{(n)}(x). \tag{5}$$

If we neglect the last integration in the right-hand side of Eq. (5), the error bound is given by

$$\left|\underbrace{\int_a^x dx \cdots \int_a^x dx}_{n} f^{(n)}(x)\right| \le \max_{[a,x]}\left|f^{(n)}(x)\right| \underbrace{\int_a^x dx \cdots \int_a^x dx}_{n} \cdot 1 = \max_{[a,x]}\left|f^{(n)}(x)\right|(x-a)^n. \tag{6}$$