**47. Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement.

48. A real number x is called an **upper bound** of a set S of real numbers if x is greater than or equal to every member of S. The real number x is called the **least upper bound** of a set S of real numbers if x is an upper bound of S and x is less than or equal to every upper bound of S; if the least upper bound of a set S exists, it is unique.
   a) Using quantifiers, express the fact that x is an upper bound of S.
   b) Using quantifiers, express the fact that x is the least upper bound of S.

*49. Express the quantification $\exists! x P(x)$ using universal quantifications, existential quantifications, and logical operators.

The statement $\lim_{n\to\infty} a_n = L$ means that for every positive real number ϵ there is a positive integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

50. (Calculus required) Use quantifiers to express the statement that $\lim_{n\to\infty} a_n = L$.

51. (Calculus required) Use quantifiers to express the statement that $\lim_{n\to\infty} a_n$ does not exist.

52. (Calculus required) Use quantifiers to express this definition: A sequence $\{a_n\}$ is a Cauchy sequence if for every real number $\epsilon > 0$ there exists a positive integer N such that $|a_m - a_n| < \epsilon$ for every pair of positive integers m and n with $m > N$ and $n > N$.

53. (Calculus required) Use quantifiers and logical connectives to express this definition: A number L is the **limit superior** of a sequence $\{a_n\}$ if for every real number $\epsilon > 0$, $a_n > L - \epsilon$ for infinitely many n and $a_n > L + \epsilon$ for only finitely many n.

1.5 Methods of Proof

INTRODUCTION

Two important questions that arise in the study of mathematics are: (1) When is a mathematical argument correct? (2) What methods can be used to construct mathematical arguments? This section helps answer these questions by describing various forms of correct and incorrect mathematical arguments.

Links

A **theorem** is a statement that can be shown to be true. (Theorems are sometimes called *propositions, facts,* or *results.*) We demonstrate that a theorem is true with a sequence of statements that form an argument, called a **proof.** To construct proofs, methods are needed to derive new statements from old ones. The statements used in a proof can include **axioms** or **postulates,** which are the underlying assumptions about mathematical structures, the hypotheses of the theorem to be proved, and previously proved theorems. The **rules of inference,** which are the means used to draw conclusions from other assertions, tie together the steps of a proof.

In this section rules of inference will be discussed. This will help clarify what makes up a correct proof. Some common forms of incorrect reasoning, called **fallacies,** will also be described. Then various methods commonly used to prove theorems will be introduced.

The terms *lemma* and *corollary* are used for certain types of theorems. A **lemma** (plural **lemmas** or **lemmata**) is a simple theorem used in the proof of other theorems. Complicated proofs are usually easier to understand when they are proved using a series of lemmas, where each lemma is proved individually. A **corollary** is a proposition that can be established directly from a theorem that has been proved. A **conjecture** is a statement whose truth value is unknown. When a proof of a conjecture is found, the conjecture becomes a theorem. Many times conjectures are shown to be false, so they are not theorems.

The methods of proof discussed in this chapter are important not only because they are used to prove mathematical theorems, but also for their many applications to computer science. These applications include verifying that computer programs are correct, establishing that operating systems are secure, making inferences in the area of artificial

intelligence, showing that system specifications are consistent, and so on. Consequently, understanding the techniques used in proofs is essential both in mathematics and in computer science.

RULES OF INFERENCE

We will now introduce rules of inference for propositional logic. These rules provide the justification of the steps used to show that a conclusion follows logically from a set of hypotheses. The tautology $(p \wedge (p \rightarrow q)) \rightarrow q$ is the basis of the rule of inference called **modus ponens,** or the **law of detachment.** This tautology is written in the following way:

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Using this notation, the hypotheses are written in a column and the conclusion below a bar. (The symbol $\therefore$ denotes "therefore.") Modus ponens states that if both an implication and its hypothesis are known to be true, then the conclusion of this implication is true.

Extra Examples

EXAMPLE 1 Suppose that the implication "if it snows today, then we will go skiing" and its hypothesis "it is snowing today," are true. Then, by modus ponens, it follows that the conclusion of the implication, "we will go skiing," is true. ◀

EXAMPLE 2 Assume that the implication "if n is greater than 3, then n^2 is greater than 9" is true. Consequently, if n is greater than 3, then, by modus ponens, it follows that n^2 is greater than 9. ◀

Table 1 lists some important rules of inference. The verifications of these rules of inference can be found as exercises in Section 1.2. Here are some examples of arguments using these rules of inference.

EXAMPLE 3 State which rule of inference is the basis of the following argument: "It is below freezing now. Therefore, it is either below freezing or raining now."

Solution: Let p be the proposition "It is below freezing now" and q the proposition "It is raining now." Then this argument is of the form

$$\begin{array}{l} p \\ \underline{} \\ \therefore p \vee q \end{array}$$

This is an argument that uses the addition rule. ◀

EXAMPLE 4 State which rule of inference is the basis of the following argument: "It is below freezing and raining now. Therefore, it is below freezing now."

Solution: Let p be the proposition "It is below freezing now," and let q be the proposition "It is raining now." This argument is of the form

$$\begin{array}{l} p \wedge q \\ \underline{} \\ \therefore p \end{array}$$

This argument uses the simplification rule. ◀

EXAMPLE 5 State which rule of inference is used in the argument:

> If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$	$p \rightarrow (p \vee q)$	Addition
$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$	$(p \wedge q) \rightarrow p$	Simplification
$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore\ q \end{array}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore\ \neg p \end{array}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

Solution: Let p be the proposition "It is raining today," let q be the proposition "We will not have a barbecue today," and let r be the proposition "We will have a barbecue tomorrow." Then this argument is of the form

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore\ p \rightarrow r \end{array}$$

Hence, this argument is a hypothetical syllogism. ◀

VALID ARGUMENTS

An argument form is called **valid** if whenever all the hypotheses are true, the conclusion is also true. Consequently, showing that q logically follows from the hypotheses $p_1, p_2, \ldots, p_n$ is the same as showing that the implication

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

is true. When all propositions used in a valid argument are true, it leads to a correct conclusion. However, a valid argument can lead to an incorrect conclusion if one or more false propositions are used within the argument. For example,

"If $\sqrt{2} > \frac{1}{2}$, then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Consequently, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$."

is a valid argument form based on modus ponens. However, the conclusion of this argument is false, because $2 < \frac{9}{4}$. The false proposition "$\sqrt{2} > \frac{3}{2}$" has been used in the argument, which means that the conclusion of the argument may be false.

Extra Examples

When there are many premises, several rules of inference are often needed to show that an argument is valid. This is illustrated by the following examples, where the steps of arguments are displayed step by step, with the reason for each step explicitly stated. These examples also show how arguments in English can be analyzed using rules of inference.

EXAMPLE 6 Show that the hypotheses "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the hypotheses become $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t.

We construct an argument to show that our hypotheses lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification using Step 1
3. $r \rightarrow p$	Hypothesis
4. $\neg r$	Modus tollens using Steps 2 and 3
5. $\neg r \rightarrow s$	Hypothesis
6. s	Modus ponens using Steps 4 and 5
7. $s \rightarrow t$	Hypothesis
8. t	Modus ponens using Steps 6 and 7

◀

EXAMPLE 7 Show that the hypotheses "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the hypotheses are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$. The desired conclusion is $\neg q \rightarrow s$.

This argument form shows that our hypotheses lead to the desired conclusion.

Step	Reason
1. $p \rightarrow q$	Hypothesis
2. $\neg q \rightarrow \neg p$	Contrapositive of Step 1
3. $\neg p \rightarrow r$	Hypothesis
4. $\neg q \rightarrow r$	Hypothetical syllogism using Steps 2 and 3
5. $r \rightarrow s$	Hypothesis
6. $\neg q \rightarrow s$	Hypothetical syllogism using Steps 4 and 5

◀

RESOLUTION

Links

Computer programs have been developed to automate the task of reasoning and proving theorems. Many of these programs make use of a rule of inference known as **resolution.** This rule of inference is based on the tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r).$$

(The verification that this is a tautology was addressed in Exercise 28 in Section 1.2.) The final disjunction in the resolution rule, $q \vee r$, is called the **resolvent.** When we let $q = r$ in this tautology, we obtain $(p \vee q) \wedge (\neg p \vee q) \rightarrow q$. Furthermore, when we let $r = \mathbf{F}$, we obtain $(p \vee q) \wedge (\neg p) \rightarrow q$ (because $q \vee \mathbf{F} \equiv q$), which is the tautology on which the rule of disjunctive syllogism is based.

EXAMPLE 8 Use resolution to show that the hypotheses "Jasmine is skiing or it is not snowing" and "It is snowing or Bart is playing hockey" imply that "Jasmine is skiing or Bart is playing hockey."

Extra Examples

Solution: Let p be the proposition "It is snowing," q the proposition "Jasmine is skiing," and r the proposition "Bart is playing hockey." We can represent the hypotheses as $\neg p \vee q$ and $p \vee r$, respectively. Using resolution, the proposition $q \vee r$, "Jasmine is skiing or Bart is playing hockey," follows. ◀

Resolution plays an important role in programming languages based on the rules of logic, such as Prolog (where resolution rules for quantified statements are applied). Furthermore, it can be used to build automatic theorem proving systems. To construct proofs in propositional logic using resolution as the only rule of inference, the hypotheses and the conclusion must be expressed as **clauses,** where a clause is a disjunction of variables or negations of these variables. We can replace a statement in propositional logic that is not a clause by one or more equivalent statements that are clauses. For example, suppose we have a statement of the form $p \vee (q \wedge r)$. Because $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, we can replace the single statement $p \vee (q \wedge r)$ by two statements $p \vee q$ and $p \vee r$, each of which is a clause. We can replace a statement of the form $\neg(p \vee q)$ by the two statements $\neg p$ and $\neg q$ because De Morgan's law tells us that $\neg(p \vee q) \equiv \neg p \wedge \neg q$. We can also replace an implication $p \rightarrow q$ with the equivalent disjunction $\neg p \vee q$.

EXAMPLE 9 Show that the hypotheses $(p \wedge q) \vee r$ and $r \rightarrow s$ imply the conclusion $p \vee s$.

Solution: We can rewrite the hypothesis $(p \wedge q) \vee r$ as two clauses, $p \vee r$ and $q \vee r$. We can also replace $r \rightarrow s$ by the equivalent clause $\neg r \vee s$. Using the two clauses $p \vee r$ and $\neg r \vee s$, we can use resolution to conclude $p \vee s$. ◀

FALLACIES

Links

Several common fallacies arise in incorrect arguments. These fallacies resemble rules of inference but are based on contingencies rather than tautologies. These are discussed here to show the distinction between correct and incorrect reasoning.

The proposition $[(p \rightarrow q) \wedge q] \rightarrow p$ is not a tautology, since it is false when p is false and q is true. However, there are many incorrect arguments that treat this as a tautology. This type of incorrect reasoning is called the **fallacy of affirming the conclusion.**

EXAMPLE 10 Is the following argument valid?

If you do every problem in this book, then you will learn discrete mathematics. You learned discrete mathematics.

Therefore, you did every problem in this book.

Solution: Let p be the proposition "You did every problem in this book." Let q be the proposition "You learned discrete mathematics." Then this argument is of the form: if $p \rightarrow q$ and q, then p. This is an example of an incorrect argument using the fallacy of affirming the conclusion. Indeed, it is possible for you to learn discrete mathematics in some way other than by doing every problem in this book. (You may learn discrete mathematics by reading, listening to lectures, doing some but not all the problems in this book, and so on.) ◀

The proposition $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is not a tautology, since it is false when p is false and q is true. Many incorrect arguments use this incorrectly as a rule of inference. This type of incorrect reasoning is called the **fallacy of denying the hypothesis.**

EXAMPLE 11 Let p and q be as in Example 10. If the implication $p \rightarrow q$ is true, and $\neg p$ is true, is it correct to conclude that $\neg q$ is true? In other words, is it correct to assume that you did not learn discrete mathematics if you did not do every problem in the book, assuming that if you do every problem in this book, then you will learn discrete mathematics?

Solution: It is possible that you learned discrete mathematics even if you did not do every problem in this book. This incorrect argument is of the form $p \rightarrow q$ and $\neg p$ imply $\neg q$, which is an example of the fallacy of denying the hypothesis. ◀

RULES OF INFERENCE FOR QUANTIFIED STATEMENTS

We discussed rules of inference for propositions. We will now describe some important rules of inference for statements involving quantifiers. These rules of inference are used extensively in mathematical arguments, often without being explicitly mentioned.

Universal instantiation is the rule of inference used to conclude that $P(c)$ is true, where c is a particular member of the universe of discourse, given the premise $\forall x\, P(x)$. Universal instantiation is used when we conclude from the statement "All women are wise" that "Lisa is wise," where Lisa is a member of the universe of discourse of all women.

Universal generalization is the rule of inference that states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the universe of discourse. Universal generalization is used when we show that $\forall x\, P(x)$ is true by taking an arbitrary element c from the universe of discourse and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the universe of discourse. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly.

Existential instantiation is the rule that allows us to conclude that there is an element c in the universe of discourse for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Since it exists, we may give it a name (c) and continue our argument.

TABLE 2 Rules of Inference for Quantified Statements.	
Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the universe of discourse for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

We summarize these rules of inference in Table 2. We will illustrate how one of these rules of inference for quantified statements is used in Example 12.

EXAMPLE 12 Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Extra Examples

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3) ◀

EXAMPLE 13 Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise

Step	Reason
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8) ◀

Remark: Mathematical arguments often include steps where both a rule of inference for propositions and a rule of inference for quantifiers are used. For example, universal instantiation and modus ponens are often used together. When these rules of inference are combined, the hypothesis $\forall x(P(x) \rightarrow Q(x))$ and $P(c)$, where c is a member of the universe of discourse, show that the conclusion $Q(c)$ is true.

Remark: Many theorems in mathematics state that a property holds for all elements in a particular set, such as the set of integers or the set of real numbers. Although the precise statement of such theorems needs to include a universal quantifier, the standard convention in mathematics is to omit it. For example, the statement "If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$" really means "For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$." Furthermore, when theorems of this type are proved, the law of universal generalization is often used without explicit mention. The first step of the proof usually involves selecting a general element of the universe of discourse. Subsequent steps show that this element has the property in question. Universal generalization implies that the theorem holds for all members of the universe of discourse.

Extra Examples

In our subsequent discussions, we will follow the usual conventions and not explicitly mention the use of universal quantification and universal generalization. However, you should always understand when this rule of inference is being implicitly applied.

METHODS OF PROVING THEOREMS

Assessment

Proving theorems can be difficult. We need all the ammunition that is available to help us prove different results. We now introduce a battery of different proof methods. These methods should become part of your repertoire for proving theorems. Because many theorems are implications, the techniques for proving implications are important. Recall that $p \rightarrow q$ is true unless p is true but q is false. Note that when the statement $p \rightarrow q$ is proved, it need only be shown that q is true if p is true; it is *not* usually the case that q is proved to be true. The following discussion will give the most common techniques for proving implications.

DIRECT PROOFS The implication $p \rightarrow q$ can be proved by showing that if p is true, then q must also be true. This shows that the combination p true and q false never occurs. A proof of this kind is called a **direct proof.** To carry out such a proof, assume that p is true and use rules of inference and theorems already proved to show that q must also be true.

Extra Examples

Before we give an example of a direct proof, we need a definition.

DEFINITION 1 The integer n is *even* if there exists an integer k such that $n = 2k$ and it is *odd* if there exists an integer k such that $n = 2k + 1$. (Note that an integer is either even or odd.)

EXAMPLE 14 Give a direct proof of the theorem "If n is an odd integer, then n^2 is an odd integer."

Solution: Assume that the hypothesis of this implication is true, namely, suppose that n is odd. Then $n = 2k + 1$, where k is an integer. It follows that $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Therefore, n^2 is an odd integer (it is one more than twice an integer). ◀

Extra Examples

INDIRECT PROOFS Since the implication $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$, the implication $p \rightarrow q$ can be proved by showing that its contrapositive, $\neg q \rightarrow \neg p$, is true. This related implication is usually proved directly, but any proof technique can be used. An argument of this type is called an **indirect proof.**

EXAMPLE 15 Give an indirect proof of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: Assume that the conclusion of this implication is false; namely, assume that n is even. Then $n = 2k$ for some integer k. It follows that $3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$, so $3n + 2$ is even (since it is a multiple of 2) and therefore not odd. Because the negation of the conclusion of the implication implies that the hypothesis is false, the original implication is true. ◀

VACUOUS AND TRIVIAL PROOFS Suppose that the hypothesis p of an implication $p \rightarrow q$ is false. Then the implication $p \rightarrow q$ is true, because the statement has the form $\mathbf{F} \rightarrow \mathbf{T}$ or $\mathbf{F} \rightarrow \mathbf{F}$, and hence is true. Consequently, if it can be shown that p is false, then a proof, called a **vacuous proof,** of the implication $p \rightarrow q$ can be given. Vacuous proofs are often used to establish special cases of theorems that state that an implication is true for all positive integers [i.e., a theorem of the kind $\forall n\, P(n)$ where $P(n)$ is a propositional function]. Proof techniques for theorems of this kind will be discussed in Section 3.3.

EXAMPLE 16 Show that the proposition $P(0)$ is true where $P(n)$ is the propositional function "If $n > 1$, then $n^2 > n$."

Solution: Note that the proposition $P(0)$ is the implication "If $0 > 1$, then $0^2 > 0$." Since the hypothesis $0 > 1$ is false, the implication $P(0)$ is automatically true. ◀

Remark: The fact that the conclusion of this implication, $0^2 > 0$, is false is irrelevant to the truth value of the implication, because an implication with a false hypothesis is guaranteed to be true.

Suppose that the conclusion q of an implication $p \rightarrow q$ is true. Then $p \rightarrow q$ is true, since the statement has the form $\mathbf{T} \rightarrow \mathbf{T}$ or $\mathbf{F} \rightarrow \mathbf{T}$, which are true. Hence, if it can be shown that q is true, then a proof, called a **trivial proof,** of $p \rightarrow q$ can be given. Trivial proofs are often important when special cases of theorems are proved (see the discussion of proof by cases) and in mathematical induction, which is a proof technique discussed in Section 3.3.

EXAMPLE 17 Let $P(n)$ be "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$." Show that the proposition $P(0)$ is true.

Solution: The proposition $P(0)$ is "If $a \geq b$, then $a^0 \geq b^0$." Since $a^0 = b^0 = 1$, the conclusion of $P(0)$ is true. Hence, $P(0)$ is true. This is an example of a trivial proof. Note that the hypothesis, which is the statement "$a \geq b$," was not needed in this proof. ◀

A LITTLE PROOF STRATEGY We have described both direct and indirect proofs and we have provided an example of how they are used. However, when confronted with an implication to prove, which method should you use? First, quickly evaluate whether a direct proof looks promising. Begin by expanding the definitions in the hypotheses. Then begin to reason using them, together with axioms and available theorems. If a direct proof does not seem to go anywhere, try the same thing with an indirect proof. Recall that in an indirect proof you assume that the conclusion of the implication is false and use a direct proof to show this implies that the hypothesis must be false. Sometimes when there is no obvious way to approach a direct proof, an indirect proof works nicely. We illustrate this strategy in Examples 18 and 19.

Extra Examples

Before we present our next example, we need a definition.

DEFINITION 2 The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational.*

EXAMPLE 18 Prove that the sum of two rational numbers is rational.

Solution: We first attempt a direct proof. To begin, suppose that r and s are rational numbers. From the definition of a rational number, it follows that there are integers p and q, with $q \neq 0$, such that $r = p/q$, and integers t and u, with $u \neq 0$, such that $s = t/u$. Can we use this information to show that $r + s$ is rational? The obvious next step is to add $r = p/q$ and $s = t/u$, to obtain

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu}.$$

Because $q \neq 0$ and $u \neq 0$, it follows that $qu \neq 0$. Consequently, we have expressed $r + s$ as the ratio of two integers, $pu + qt$ and qu, where $qu \neq 0$. This means that $r + s$ is rational. Our attempt to find a direct proof succeeded. ◀

EXAMPLE 19 Prove that if n is an integer and n^2 is odd, then n is odd.

Solution: We first attempt a direct proof. Suppose that n is an integer and n^2 is odd. Then, there exists an integer k such that $n^2 = 2k + 1$. Can we use this information to show that n is odd? There seems to be no obvious approach to show that n is odd because solving for n produces the equation $n = \pm\sqrt{2k + 1}$, which is not terribly useful.

Because this attempt to use a direct proof did not bear fruit, we next attempt an indirect proof. We take as our hypothesis the statement that n is not odd. Because every integer is odd or even, this means that n is even. This implies that there exists an integer k such that $n = 2k$. To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd, that is, that n^2 is even. Can we use the equation $n = 2k$ to achieve this? By squaring both sides of this equation, we obtain $n^2 = 4k^2 = 2(2k^2)$, which implies that n^2 is also even since $n^2 = 2t$, where $t = 2k^2$. Our attempt to find an indirect proof succeeded. ◀

PROOFS BY CONTRADICTION There are other approaches we can use when neither a direct proof nor an indirect proof succeeds. We now introduce several additional proof techniques.

Suppose that a contradiction q can be found so that $\neg p \rightarrow q$ is true, that is, $\neg p \rightarrow \mathbf{F}$ is true. Then the proposition $\neg p$ must be false. Consequently, p must be true. This technique can be used when a contradiction, such as $r \wedge \neg r$, can be found so that it is possible to show that the implication $\neg p \rightarrow (r \wedge \neg r)$ is true. An argument of this type is called a **proof by contradiction.**

We provide three examples of proof by contradiction. The first is an example of an application of the pigeonhole principle, a combinatorial technique which we will cover in depth in Section 4.2.

EXAMPLE 20 Show that at least four of any 22 days must fall on the same day of the week.

Extra Examples

Solution: Let p be the proposition "At least four of the 22 chosen days are the same day of the week." Suppose that $\neg p$ is true. Then at most three of the 22 days are the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen since three is the most days chosen that could be a particular day of the week. This is a contradiction. ◀

EXAMPLE 21 Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution: Let p be the proposition "$\sqrt{2}$ is irrational." Suppose that $\neg p$ is true. Then $\sqrt{2}$ is rational. We will show that this leads to a contradiction. Under the assumption that $\sqrt{2}$ is rational, there exist integers a and b with $\sqrt{2} = a/b$, where a and b have no common factors (so that the fraction a/b is in lowest terms). Since $\sqrt{2} = a/b$, when both sides of this equation are squared, it follows that

$$2 = a^2/b^2.$$

Hence,

$$2b^2 = a^2.$$

This means that a^2 is even, implying that a is even. Furthermore, since a is even, $a = 2c$ for some integer c. Thus

$$2b^2 = 4c^2,$$

so

$$b^2 = 2c^2.$$

This means that b^2 is even. Hence, b must be even as well.

It has been shown that $\neg p$ implies that $\sqrt{2} = a/b$, where a and b have no common factors, and 2 divides a and b. This is a contradiction since we have shown that $\neg p$ implies both r and $\neg r$ where r is the statement that a and b are integers with no common factors. Hence, $\neg p$ is false, so that p: "$\sqrt{2}$ is irrational" is true. ◀

An indirect proof of an implication can be rewritten as a proof by contradiction. In an indirect proof we show that $p \rightarrow q$ is true by using a direct proof to show that $\neg q \rightarrow \neg p$ is true. That is, in an indirect proof of $p \rightarrow q$ we assume that $\neg q$ is true and show that $\neg p$ must also be true. To rewrite an indirect proof of $p \rightarrow q$ as a proof by contradiction,

we suppose that both p and $\neg q$ are true. Then we use the steps from the direct proof of $\neg q \rightarrow \neg p$ to show that $\neg p$ must also be true. This leads to the contradiction $p \wedge \neg p$, completing the proof by contradiction. Example 22 illustrates how an indirect proof of an implication can be rewritten as a proof by contradiction.

EXAMPLE 22 Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

Solution: We assume that $3n + 2$ is odd and that n is not odd, so that n is even. Following the same steps as in the solution of Example 15 (an indirect proof of this theorem), we can show that if n is even, then $3n + 2$ is even. This contradicts the assumption that $3n + 2$ is odd, completing the proof. ◀

PROOF BY CASES To prove an implication of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \rightarrow q] \leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \cdots \wedge (p_n \rightarrow q)]$$

can be used as a rule of inference. This shows that the original implication with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \ldots, p_n$ can be proved by proving each of the n implications $p_i \rightarrow q, i = 1, 2, \ldots, n$, individually. Such an argument is called a **proof by cases.** Sometimes to prove that an implication $p \rightarrow q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the implication, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent. Consider Example 23.

Extra Examples

EXAMPLE 23 Use a proof by cases to show that $|xy| = |x||y|$, where x and y are real numbers. (Recall that $|x|$, the absolute value of x, equals x when $x \geq 0$ and equals $-x$ when $x \leq 0$.)

Solution: Let p be "x and y are real numbers" and let q be "$|xy| = |x||y|$." Note that p is equivalent to $p_1 \vee p_2 \vee p_3 \vee p_4$, where p_1 is "$x \geq 0 \wedge y \geq 0$," p_2 is "$x \geq 0 \wedge y < 0$," p_3 is "$x < 0 \wedge y \geq 0$," and p_4 is "$x < 0 \wedge y < 0$." Hence, to show that $p \rightarrow q$, we can show that $p_1 \rightarrow q$, $p_2 \rightarrow q$, $p_3 \rightarrow q$, and $p_4 \rightarrow q$. (We have used these four cases because we can remove the absolute value signs by making the appropriate choice of signs within each case.)

We see that $p_1 \rightarrow q$ because $xy \geq 0$ when $x \geq 0$ and $y \geq 0$, so that $|xy| = xy = |x||y|$.

To see that $p_2 \rightarrow q$, note that if $x \geq 0$ and $y < 0$, then $xy \leq 0$, so that $|xy| = -xy = x(-y) = |x||y|$. (Here, because $y < 0$, we have $|y| = -y$.)

To see that $p_3 \rightarrow q$, we follow the same reasoning as the previous case with the roles of x and y reversed.

To see that $p_4 \rightarrow q$, note that when $x < 0$ and $y < 0$, it follows that $xy > 0$. Hence $|xy| = xy = (-x)(-y) = |x||y|$. This completes the proof. ◀

PROOFS OF EQUIVALENCE To prove a theorem that is a biconditional, that is, one that is a statement of the form $p \leftrightarrow q$ where p and q are propositions, the tautology

$$(p \leftrightarrow q) \leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow p)]$$

can be used. That is, the proposition "p if and only if q" can be proved if both the implications "if p, then q" and "if q, then p" are proved.

EXAMPLE 24 Prove the theorem "The integer n is odd if and only if n^2 is odd."

Solution: This theorem has the form "p if and only if q," where p is "n is odd" and q is "n^2 is odd." To prove this theorem, we need to show that $p \rightarrow q$ and $q \rightarrow p$ are true.

Extra Examples

We have already shown (in Example 14) that $p \rightarrow q$ is true and (in Example 19) that $q \rightarrow p$ is true.

Since we have shown that both $p \rightarrow q$ and $q \rightarrow p$ are true, we have shown that the theorem is true. ◀

Sometimes a theorem states that several propositions are equivalent. Such a theorem states that propositions $p_1, p_2, p_3, \ldots, p_n$ are equivalent. This can be written as

$$p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n,$$

which states that all n propositions have the same truth values, and consequently, that for all i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$, p_i and p_j are equivalent. One way to prove these mutually equivalent is to use the tautology

$$[p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n] \leftrightarrow [(p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_3) \wedge \cdots \wedge (p_n \rightarrow p_1)].$$

This shows that if the implications $p_1 \rightarrow p_2, p_2 \rightarrow p_3, \ldots, p_n \rightarrow p_1$ can be shown to be true, then the propositions $p_1, p_2, \ldots, p_n$ are all equivalent.

This is much more efficient than proving that $p_i \rightarrow p_j$ for all $i \neq j$ with $1 \leq i \leq n$ and $1 \leq j \leq n$.

When we prove that a group of statements are equivalent, we can establish any chain of implications we choose as long as it is possible to work through the chain to go from any one of these statements to any other statement. For example, we can show that p_1, p_2, and p_3 are equivalent by showing that $p_1 \rightarrow p_3$, $p_3 \rightarrow p_2$, and $p_2 \rightarrow p_1$.

EXAMPLE 25 Show that these statements are equivalent:

p_1: n is an even integer.
p_2: $n - 1$ is an odd integer.
p_3: n^2 is an even integer.

Solution: We will show that these three statements are equivalent by showing that the implications $p_1 \rightarrow p_2$, $p_2 \rightarrow p_3$, and $p_3 \rightarrow p_1$ are true.

We use a direct proof to show that $p_1 \rightarrow p_2$. Suppose that n is even. Then $n = 2k$ for some integer k. Consequently, $n - 1 = 2k - 1 = 2(k - 1) + 1$. This means that $n - 1$ is odd since it is of the form $2m + 1$, where m is the integer $k - 1$.

We also use a direct proof to show that $p_2 \rightarrow p_3$. Now suppose $n - 1$ is odd. Then $n - 1 = 2k + 1$ for some integer k. Hence, $n = 2k + 2$ so that $n^2 = (2k + 2)^2 = 4k^2 + 8k + 4 = 2(2k^2 + 4k + 2)$. This means that n^2 is twice the integer $2k^2 + 4k + 2$, and hence is even.

To prove $p_3 \rightarrow p_1$, we use an indirect proof. That is, we prove that if n is not even, then n^2 is not even. This is the same as proving that if n is odd, then n^2 is odd, which we have already done in Example 14. This completes the proof. ◀

THEOREMS AND QUANTIFIERS

Many theorems are stated as propositions that involve quantifiers. A variety of methods are used to prove theorems that are quantifications. We will describe some of the most important of these here.

EXISTENCE PROOFS Many theorems are assertions that objects of a particular type exist. A theorem of this type is a proposition of the form $\exists x\, P(x)$, where P is a predicate. A proof of a proposition of the form $\exists x\, P(x)$ is called an **existence proof.** There are several ways to prove a theorem of this type. Sometimes an existence proof of $\exists x\, P(x)$ can be given by finding an element a such that $P(a)$ is true. Such an existence proof is called **constructive.** It is also possible to give an existence proof that is **nonconstructive;** that is, we do not find an element a such that $P(a)$ is true, but rather prove that $\exists x\, P(x)$ is true in some other way. One common method of giving a nonconstructive existence proof is to use proof by contradiction and show that the negation of the existential quantification implies a contradiction. The concept of a constructive existence proof is illustrated by Example 26.

Extra Examples

EXAMPLE 26 **A Constructive Existence Proof.** Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Solution: After considerable computation (such as a computer search) we find that

$$1729 = 10^3 + 9^3 = 12^3 + 1^3.$$

Because we have displayed a positive integer that can be written as the sum of cubes in two different ways, we are done. ◀

EXAMPLE 27 **A Nonconstructive Existence Proof.** Show that there exist irrational numbers x and y such that x^y is rational.

Solution: By Example 21 we know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$. If it is rational, we have two irrational numbers x and y with x^y rational, namely, $x = \sqrt{2}$ and $y = \sqrt{2}$. On the other hand if $\sqrt{2}^{\sqrt{2}}$ is irrational, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ so that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^2 = 2$.

This proof is an example of a nonconstructive existence proof because we have not found irrational numbers x and y such that x^y is rational. Rather, we have shown that either the pair $x = \sqrt{2}, y = \sqrt{2}$ or the pair $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$ have the desired property, but we do not know which of these two pairs works! ◀

UNIQUENESS PROOFS Some theorems assert the existence of a unique element with a particular property. In other words, these theorems assert that there is exactly one element with this property. To prove a statement of this type we need to show that an element with this property exists and that no other element has this property. The two parts of a uniqueness proof are:

Existence: We show that an element x with the desired property exists.
Uniqueness: We show that if $y \neq x$, then y does not have the desired property.

HISTORICAL NOTE The English mathematician G. H. Hardy, when visiting the ailing Indian prodigy Ramanujan in the hospital, remarked that 1729, the number of the cab he took, was rather dull. Ramanujan replied "No, it is a very interesting number; it is the smallest number expressible as the sum of cubes in two different ways." (See the Supplementary Exercises in Chapter 3 for biographies of Hardy and Ramanujan.)

Remark: Showing that there is a unique element x such that $P(x)$ is the same as proving the statement $\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$.

EXAMPLE 28 Show that every integer has a unique additive inverse. That is, show that if p is an integer, then there exists a unique integer q such that $p + q = 0$.

Extra Examples

Solution: If p is an integer, we find that $p+q = 0$ when $q = -p$ and q is also an integer. Consequently, there exists an integer q such that $p + q = 0$.

To show that given the integer p, the integer q with $p + q = 0$ is unique, suppose that r is an integer with $r \neq q$ such that $p + r = 0$. Then $p + q = p + r$. By subtracting p from both sides of the equation, it follows that $q = r$, which contradicts our assumption that $q \neq r$. Consequently, there is a unique integer q such that $p + q = 0$. ◀

COUNTEREXAMPLES In Section 1.3 we mentioned that we can show that a statement of the form $\forall x P(x)$ is false if we can find a counterexample, that is, an example x for which $P(x)$ is false. When we are presented with a statement of the form $\forall x P(x)$, either which we believe to be false or which has resisted all attempts to find a proof, we look for a counterexample. We illustrate the hunt for a counterexample in Example 29.

Extra Examples

EXAMPLE 29 Show that the statement "Every positive integer is the sum of the squares of three integers" is false.

Solution: We can show that this statement is false if we can find a counterexample. That is, the statement is false if we can show that there is a particular integer that is not the sum of the squares of three integers. To look for a counterexample, we try to write successive positive integers as a sum of three squares. We find that $1 = 0^2 + 0^2 + 1^2$, $2 = 0^2 + 1^2 + 1^2$, $3 = 1^2 + 1^2 + 1^2$, $4 = 0^2 + 0^2 + 2^2$, $5 = 0^2 + 1^2 + 2^2$, $6 = 1^2 + 1^2 + 2^2$, but we cannot find a way to write 7 as the sum of three squares. To show that there are not three squares that add up to 7, we note that the only possible squares we can use are those not exceeding 7, namely, 0, 1, and 4. Since no three terms where each term is 0, 1, or 4 add up to 7, it follows that 7 is a counterexample. We conclude that the statement "Every positive integer is the sum of the squares of three integers" is false. ◀

Links

A common error is to assume that one or more examples establish the truth of a statement. No matter how many examples there are where $P(x)$ is true, the universal quantification $\forall x P(x)$ may still be false. Consider Example 30.

EXAMPLE 30 Is it true that every positive integer is the sum of 18 fourth powers of integers? That is, is the statement $\forall n P(n)$ a theorem where $P(n)$ is the statement "n can be written as the sum of 18 fourth powers of integers" and the universe of discourse consists of all positive integers?

Solution: To determine whether n can be written as the sum of 18 fourth powers of integers, we might begin by examining whether n is the sum of 18 fourth powers of integers for the smallest positive integers. Because the fourth powers of integers are $0, 1, 16, 81, \ldots$, if we can select 18 terms from these numbers that add up to n, then n is the sum of 18 fourth powers. We can show that all positive integers up to 78 can be written as

the sum of 18 fourth powers. (The details are left to the reader.) However, if we decided this was enough checking, we would come to the wrong conclusion. It is not true that every positive integer is the sum of 18 fourth powers because 79 is not the sum of 18 fourth powers (as the reader can verify). ◀

MISTAKES IN PROOFS

There are many common errors made in constructing mathematical proofs. We will briefly describe some of these here. Among the most common errors are mistakes in arithmetic and basic algebra. Even professional mathematicians make such errors, especially when working with complicated formulas. Whenever you use such computations you should check them as carefully as possible. (You should also review any troublesome aspects of basic algebra, especially before you study Section 3.3.)

Each step of a mathematical proof needs to be correct and the conclusion needs to logically follow from the steps that precede it. Many mistakes result from the introduction of steps that do not logically follow from those that precede it. This is illustrated in Examples 31–33.

EXAMPLE 31 What is wrong with this famous supposed "proof" that $1 = 2$?

"Proof:" We use these steps, where a and b are two equal positive integers.

Step	Reason
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Solution: Every step is valid except for one, step 5 where we divided both sides by $a - b$. The error is that $a - b$ equals zero; division of both sides of an equation by the same quantity is valid as long as this quantity is not zero. ◀

EXAMPLE 32 What is wrong with this "proof"?

"Theorem:" If n^2 is positive, then n is positive.

"Proof:" Suppose that n^2 is positive. Because the implication "If n is positive, then n^2 is positive" is true, we can conclude that n is positive.

Solution: Let $P(n)$ be "n is positive" and $Q(n)$ be "n^2 is positive." Then our hypothesis is $Q(n)$. The statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \rightarrow Q(n))$. From the hypothesis $Q(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $P(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of affirming the conclusion. A counterexample is supplied by $n = -1$ for which $n^2 = 1$ is positive, but n is negative. ◀

EXAMPLE 33 What is wrong with this "proof"?

"Theorem:" If n is not positive, then n^2 is not positive. (This is the contrapositive of the "theorem" in Example 32.)

"Proof:" Suppose that n is not positive. Because the implication "If n is positive, then n^2 is positive" is true, we can conclude that n^2 is not positive.

Solution: Let $P(n)$ and $Q(n)$ be as in the solution of Example 32. Then our hypothesis is $\neg P(n)$ and the statement "If n is positive, then n^2 is positive" is the statement $\forall n(P(n) \rightarrow Q(n))$. From the hypothesis $\neg P(n)$ and the statement $\forall n(P(n) \rightarrow Q(n))$ we cannot conclude $\neg Q(n)$, because we are not using a valid rule of inference. Instead, this is an example of the fallacy of denying the hypothesis. A counterexample is supplied by $n = -1$, as in Example 32. ◀

A common error in making unwarranted assumptions occurs in proofs by cases, where not all cases are considered. This is illustrated in Example 34.

EXAMPLE 34 What is wrong with this "proof"?

"Theorem:" If x is a real number, then x^2 is a positive real number.

"Proof:" Let p_1 be "x is positive," let p_2 be "x is negative," and let q be "x^2 is positive." To show that $p_1 \rightarrow q$, note that when x is positive, x^2 is positive since it is the product of two positive numbers, x and x. To show that $p_2 \rightarrow q$, note that when x is negative, x^2 is positive since it is the product of two negative numbers, x and x. This completes the proof.

Solution: The problem with the proof we have given is that we missed the case $x = 0$. When $x = 0, x^2 = 0$ is not positive, so the supposed theorem is false. If p is "x is a real number," then we can prove results where p is the hypothesis with three cases, p_1, p_2, and p_3, where p_1 is "x is positive," p_2 is "x is negative," and p_3 is "$x = 0$" because of the equivalence $p \leftrightarrow p_1 \vee p_2 \vee p_3$. ◀

Finally, we briefly discuss a particularly nasty type of error. Many incorrect arguments are based on a fallacy called **begging the question.** This fallacy occurs when one or more steps of a proof are based on the truth of the statement being proved. In other words, this fallacy arises when a statement is proved using itself, or a statement equivalent to it. That is why this fallacy is also called **circular reasoning.**

EXAMPLE 35 Is the following argument correct? It supposedly shows that n is an even integer whenever n^2 is an even integer.

Suppose that n^2 is even. Then $n^2 = 2k$ for some integer k. Let $n = 2l$ for some integer l. This shows that n is even.

Solution: This argument is incorrect. The statement "let $n = 2l$ for some integer l" occurs in the proof. No argument has been given to show that n can be written as $2l$ for some integer l. This is circular reasoning because this statement is equivalent to the statement being proved, namely, "n is even." Of course, the result itself is correct; only the method of proof is wrong. ◀

Making mistakes in proofs is part of the learning process. When you make a mistake that someone else finds, you should carefully analyze where you went wrong and make sure that you do not make the same mistake again. Even professional mathematicians

make mistakes in proofs. More than a few incorrect proofs of important results have fooled people for many years before subtle errors in them were found.

JUST A BEGINNING

We have introduced a variety of methods for proving theorems. Observe that no algorithm for proving theorems has been given here or even mentioned. It is a deep result that no such procedure exists.

There are many theorems whose proofs are easy to find by directly working through the hypotheses and definitions of the terms of the theorem. However, it is often difficult to prove a theorem without resorting to a clever use of an indirect proof or a proof by contradiction, or some other proof technique. Constructing proofs is an art that can be learned only through experience, including writing proofs, having your proofs critiqued, reading and analyzing other proofs, and so on.

We will present a variety of proofs in the rest of this chapter and in Chapter 2 before we return to the subject of proofs. In Chapter 3 we will address some of the art and the strategy in proving theorems and in working with conjectures. We will also introduce several important proof techniques in Chapter 3, including mathematical induction, which can be used to prove results that hold for all positive integers. In Chapter 4 we will introduce the notion of combinatorial proofs.

Exercises

1. What rule of inference is used in each of these arguments?
   a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
   b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
   c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
   d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
   e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.
2. What rule of inference is used in each of these arguments?
   a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
   b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
   c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
   d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.
   e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.
3. Construct an argument using rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."
4. Construct an argument using rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."
5. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."
6. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."
7. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the