Fourier Transform of Step Function

2/25/10

Consider a step function

$$\theta(t) = \begin{cases} 1 & (t>0) \\ 0 & (t<0) \end{cases} \qquad (1)$$

and its Fourier transform defined through

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{\theta}(\omega) e^{-i\omega t} \qquad (2)$$

Then,

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty} dt\, \theta(t) e^{i\omega t}$$

$$= \lim_{\delta \to 0} \int_0^{\infty} dt\, e^{i\omega t - \delta t}$$

Here, the factor $e^{-\delta t}$ is introduced to make the integration convergent, and we take the limit $\delta \to 0$ afterward.

$$\therefore \tilde{\theta}(\omega) = \lim_{\delta \to 0} \int_0^{\infty} dt\, e^{i(\omega + i\delta)t}$$

$$= \lim_{\delta \to 0} \underbrace{\left[\frac{e^{i\omega t - \delta t}}{i(\omega + i\delta)} \right]_0^{\infty}}_{-\frac{1}{i(\omega+i\delta)}}$$

$$= \lim_{\delta \to 0} \frac{i}{\omega + i\delta} = \frac{i}{\omega + i0}$$

In summary,

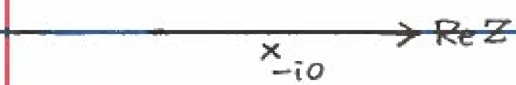
$$\boxed{\begin{aligned} \theta(t) &= \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{i}{\omega + i\delta} e^{-i\omega t} \qquad (3) \\ &= -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega + i0} \qquad (4) \end{aligned}}$$

— Contour integration

To verify Eq. (4), let us consider a contour integration

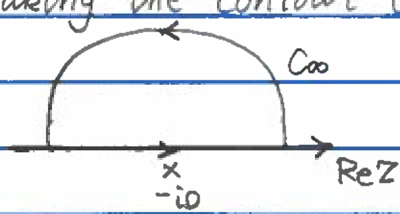
$$I = -\int_C \frac{dZ}{2\pi i} \frac{e^{-iZt}}{Z+i0} \tag{5}$$

which has a pole at $Z = -i0$.



Case 1: $t < 0$

For $Z = \omega + i\eta$, $e^{-iZt} = e^{-i(\omega+i\eta)t} = e^{-i\omega t} e^{\eta t}$. For $t<0$, therefore, e^{-iZt} exponentially vanishes at far upper plane. By taking the contour C as below, then

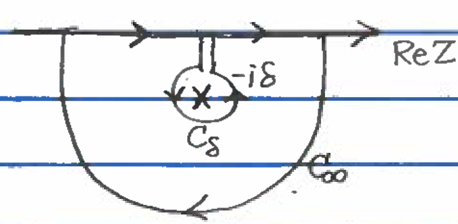


$$I = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega+i0} \cancel{- \int_{C_\infty} \frac{dZ}{2\pi i} \frac{e^{-iZt}}{Z+i0}} = 0$$

$$\therefore -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega+i0} = 0 \quad (t<0)$$

Case 2: $t > 0$

We take the following contour.



$$I = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega+i0} - \int_{C_\delta} \frac{dZ}{2\pi i} \frac{e^{-iZt}}{Z+i0} \cancel{- \int_{C_\infty} \frac{dZ}{2\pi i} \frac{e^{-iZt}}{Z+i0}}$$

For C_δ integration, we introduce a coordinate transformation

$$Z = -i\delta + re^{i\theta} \qquad (r \to 0)$$

$$\therefore dZ = ire^{i\theta} d\theta$$

Then,

$$I_{C_\delta} = -\int_0^{2\pi} \frac{\cancel{\delta}\,\cancel{re^{i\theta}}}{2\pi\cancel{\delta}} d\theta \, \frac{e^{-ire^{i\theta}+\delta t}}{\cancel{re^{i\theta}}} \quad \to 1 \quad (r \to 0,\ \delta \to 0)$$

$$= -\int_0^{2\pi} \frac{d\theta}{2\pi}$$

$$= -1$$

Therefore,

$$I = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega + i0} - 1 = 0$$

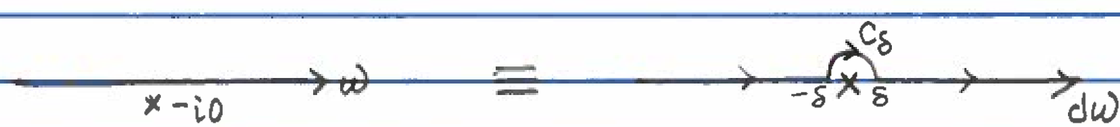
$$\therefore \ -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega + i0} = 1 \quad (t > 0) \qquad //$$

— Principal integration

Let us consider an integration

$$I = \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega + i0} \qquad (6)$$

Here, $+i0$ denotes that we avoid the singularity from above, namely



$$\therefore I = \int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega + i0} = \lim_{\delta\to 0}\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) d\omega \frac{f(\omega)}{\omega} + \int_{C_\delta} dZ \frac{f(Z)}{Z} \tag{7}$$

For the contour, we introduce a coordinate transformation,

$$Z = \delta e^{i\theta} \quad (\delta \to 0)$$

$$\therefore dZ = i\delta e^{i\theta} d\theta$$

$$\therefore \int_{C_\delta} dZ \frac{f(Z)}{Z} = \int_{\pi}^{0} i\cancel{\delta e^{i\theta}} d\theta \frac{f(\delta e^{i\theta})}{\cancel{\delta e^{i\theta}}} \quad (\delta \to 0)$$

$$= i f(0) \left[\theta\right]_{\pi}^{0}$$

$$= -i\pi f(0)$$

$$\boxed{\begin{array}{l} \therefore \displaystyle\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega + i0} = P\int_{-\infty}^{\infty} d\omega \frac{f(\omega)}{\omega} - i\pi f(0) \quad (8) \\ \text{where the principal integration is defined as} \\ \displaystyle P\int_{-\infty}^{\infty} d\omega \equiv \lim_{\delta\to 0}\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right) d\omega \quad (9) \\ \text{or} \\ \displaystyle \frac{1}{\omega + i0} = \frac{P}{\omega} - i\pi\delta(\omega) \quad (10) \end{array}}$$