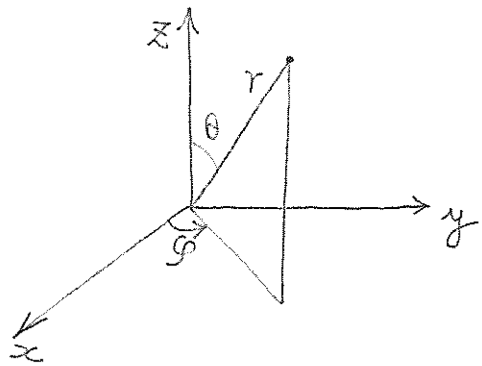
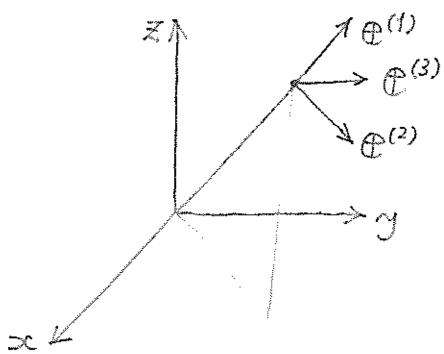
Laplacian in Spherical Coordinates

11/25/99



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad (1) \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right) \\ \varphi = \tan^{-1}(y/x) \end{cases} \quad (2)$$

Key concept = orthogonal coordinate system



$$\mathbb{x} \equiv {}^T(x_1, x_2, x_3) = {}^T(x, y, z) \quad (3)$$

$$\mathbb{q} \equiv {}^T(q_1, q_2, q_3) = {}^T(r, \theta, \varphi) \quad (4)$$

$$\mathbb{v}^{(k)} \equiv \frac{\partial \mathbb{x}}{\partial q_k} \quad (k = 1, 2, 3) \quad (5)$$

$\mathbb{v}^{(k)}$ points to the direction of $r|\theta|\varphi$ axis.

$$\mathbf{v}^{(1)} = (\partial x/\partial r,\ \partial y/\partial r,\ \partial z/\partial r)$$
$$= (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$
$$\mathbf{v}^{(2)} = (\partial x/\partial\theta,\ \partial y/\partial\theta,\ \partial z/\partial\theta)$$
$$= (r\cos\theta\cos\varphi,\ r\cos\theta\sin\varphi,\ -r\sin\theta)$$
$$\mathbf{v}^{(3)} = (\partial x/\partial\varphi,\ \partial y,\partial\varphi,\ \partial z/\partial\varphi)$$
$$= (-r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi,\ 0)$$

Transformation matrix

$$\frac{\partial \mathbf{X}}{\partial \mathbf{\Xi}} = \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \tag{6}$$

$$= \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\varphi \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial\varphi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\varphi \end{pmatrix} \tag{7}$$

$$= (\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \mathbf{v}^{(3)}) \tag{8}$$

$$= \left(\begin{array}{c|c|c} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{array}\right) \tag{9}$$

Jacobian

$$J \equiv \left|\frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)}\right| \tag{10}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix} - (-r\sin\theta) \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi \end{vmatrix}$$

$$= \cos\theta\,(r^2\sin\theta\cos\theta\cos^2\varphi + r^2\sin\theta\cos\theta\sin^2\varphi) + r\sin\theta\,(r\sin^2\theta\cos^2\varphi + r\sin^2\theta\sin^2\varphi)$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta$$

$$\therefore J = r^2\sin\theta \tag{11}$$

Orthonormal Vectors

Let's define $\boldsymbol{e}^{(k)}$ $(k=1,2,3)$ through

$$\begin{cases} \boldsymbol{v}^{(1)} = \boldsymbol{e}^{(1)} \\ \boldsymbol{v}^{(2)} = r\,\boldsymbol{e}^{(2)} \\ \boldsymbol{v}^{(3)} = r\sin\theta\,\boldsymbol{e}^{(3)} \end{cases} \tag{12}$$

or

$$\left(\boldsymbol{e}^{(1)}\ \boldsymbol{e}^{(2)}\ \boldsymbol{e}^{(3)}\right) = \left(\begin{array}{c|c|c} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{array}\right) \tag{13}$$

(Prop) $\boldsymbol{e}^{(k)}$'s are orthonormal, i.e., $\boldsymbol{e}^{(k)}\cdot\boldsymbol{e}^{(l)} = \delta_{kl}$ (14)

☺

$$|\boldsymbol{e}^{(1)}|^2 = \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$$

$$|\boldsymbol{e}^{(2)}|^2 = \cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta = \cos^2\theta + \sin^2\theta = 1$$

$$|\boldsymbol{e}^{(3)}|^2 = \sin^2\varphi + \cos^2\varphi = 1$$

$$\boldsymbol{e}^{(2)}\cdot\boldsymbol{e}^{(3)} = -\cos\theta\sin\varphi\cos\varphi + \cos\theta\sin\varphi\cos\varphi = 0$$

$$\boldsymbol{e}^{(3)}\cdot\boldsymbol{e}^{(1)} = -\sin\theta\sin\varphi\cos\varphi + \sin\theta\sin\varphi\cos\varphi = 0$$

$$\boldsymbol{e}^{(1)}\cdot\boldsymbol{e}^{(2)} = \sin\theta\cos\theta\cos^2\varphi + \sin\theta\cos\theta\sin^2\varphi - \sin\theta\cos\theta = 0 \quad //$$

Metric Factor

$$\boldsymbol{v}^{(k)} = h_k\,\boldsymbol{e}^{(k)} \qquad (k=1,2,3) \tag{15}$$

or

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta \tag{16}$$

Note that

$$\boldsymbol{e}^{(k)} = \frac{1}{h_k}\frac{\partial \boldsymbol{x}}{\partial q_k} \tag{17}$$

– Inverse

$$\mathbb{E} \equiv (\mathbb{e}^{(1)} \ \mathbb{e}^{(2)} \ \mathbb{e}^{(3)}) \quad \text{or} \quad E_{ik} = e_i^{(k)} \tag{18}$$

then

$$E^{-1}_{kj} = h_k \frac{\partial q_k}{\partial x_j} \tag{19}$$

$\because$

$$E_{ik} = \frac{1}{h_k}\frac{\partial x_i}{\partial q_k}$$

$$\sum_{k=1}^{3} \underbrace{\frac{1}{\cancel{h_k}}\frac{\partial x_i}{\partial q_k}}_{E_{ik}} \cancel{h_k} \frac{\partial q_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij} \quad //$$

Note that the inverse of an orthogonal matrix is the transpose.

$$h_k \frac{\partial q_k}{\partial x_i} = E^{-1}_{ki} = {}^{T}E_{ki} = E_{ik} = \frac{1}{h_k}\frac{\partial x_i}{\partial q_k}$$

$$\boxed{\therefore \quad \frac{1}{h_k}\frac{\partial x_i}{\partial q_k} = e_i^{(k)} = h_k \frac{\partial q_k}{\partial x_i}} \tag{20}$$

(Confirmation)

$$\frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\varphi, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\varphi, \quad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta$$

$$-\sin\theta\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\frac{z}{\sqrt{x^2+y^2+z^2}} = -\frac{z\cdot\cancel{2}x}{\cancel{2}r^3}$$

$$\cancel{\sin\theta}\frac{\partial\theta}{\partial x} = \frac{1}{r}\frac{x}{r}\frac{z}{r} = \frac{1}{r}\cancel{\sin\theta}\cos\varphi\cos\theta \qquad \therefore\ r\frac{\partial\theta}{\partial x} = \cos\varphi\cos\theta$$

similarly,

$$r\frac{\partial\theta}{\partial y} = \sin\varphi\cos\theta$$

$$-\cancel{\sin\theta}\frac{\partial\theta}{\partial z} = \frac{1}{r} - \frac{\cancel{2}z^2}{\cancel{2}r^3} = \frac{1}{r}\left[1-\left(\frac{z}{r}\right)^2\right] = \frac{1}{r}(1-\cos^2\theta) = \frac{\sin^{\cancel{2}}\theta}{r}$$

$$\therefore\ r\frac{\partial\theta}{\partial z} = -\sin\theta$$

$$\frac{1}{\cos^2\varphi}\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2} = -\frac{\cancel{r\sin\theta}\sin\varphi}{r^2\sin^2\theta\cancel{\cos^2\varphi}} \quad \therefore\ r\sin\theta\frac{\partial\varphi}{\partial x} = -\sin\varphi$$

$$\frac{1}{\cos^2\varphi}\frac{\partial\varphi}{\partial y} = \frac{1}{x} = \frac{1}{r\sin\theta\cancel{\cos\varphi}} \quad \therefore\ r\sin\theta\frac{\partial\varphi}{\partial y} = \cos\varphi$$

$$\frac{\partial\varphi}{\partial z} = 0$$

In summary,

$$h_k\frac{\partial q_k}{\partial x_i} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y & \partial r/\partial z \\ r\partial\theta/\partial x & r\partial\theta/\partial y & r\partial\theta/\partial z \\ r\sin\theta\partial\varphi/\partial x & r\sin\theta\partial\varphi/\partial y & r\sin\theta\partial\varphi/\partial z \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}$$

$$= {}^T(\mathbf{e}^{(1)}\ \ \mathbf{e}^{(2)}\ \ \mathbf{e}^{(3)})$$

$$= {}^T E \qquad //$$

— Orthonormality

$\frac{1}{h_k h_l}$ but OK because of δ_{kl} $\quad$ $h_k h_l$

$$\boxed{\frac{1}{h_k^2}\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_l} = h_k^2 \sum_i \frac{\partial q_k}{\partial x_i}\frac{\partial q_l}{\partial x_i} \left(= \mathbf{e}^{(k)}\cdot\mathbf{e}^{(l)}\right) = \delta_{kl}} \qquad (21)$$

— Completeness

$$\sum_k e_i^{(k)} e_j^{(k)} = \sum_k E_{ik} E_{jk} = \sum_k E_{ik} \underbrace{{}^{T}E_{kj}}_{E^{-1}_{kj}} = \delta_{ij}$$

$$\boxed{\therefore \frac{1}{h_k^2}\sum_k \frac{\partial x_i}{\partial q_k}\frac{\partial x_j}{\partial q_k} = h_k^2 \sum_k \frac{\partial q_k}{\partial x_i}\frac{\partial q_k}{\partial x_j} = \sum_k e_i^{(k)} e_j^{(k)} = \delta_{ij}} \qquad (22)$$

(Lemma)

$$\frac{1}{J}\frac{\partial J}{\partial q_k} = \sum_l \frac{1}{h_l}\frac{\partial h_l}{\partial q_k} \tag{23}$$

$\because$

$$J = \det\left(\frac{\partial x_i}{\partial q_k}\right) = h_1 h_2 h_3$$

$$\frac{\partial J}{\partial q_k} = \frac{\partial h_1}{\partial q_k} h_2 h_3 + h_1 \frac{\partial h_2}{\partial q_k} h_3 + h_1 h_2 \frac{\partial h_3}{\partial q_k} = \underbrace{h_1 h_2 h_3}_{J} \sum_l \frac{1}{h_l}\frac{\partial h_l}{\partial q_k} \quad //$$

(Lemma) Contracted derivatives of directional vectors

$$\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial q_k} = h_k \frac{\partial h_k}{\partial q_k} \tag{24}$$

$$\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial^2 x_k}{\partial q_k \partial q_m} = -\sum_i \frac{\partial^2 x_i}{\partial q_k^2}\frac{\partial x_i}{\partial q_m} = h_k \frac{\partial h_k}{\partial q_m} \quad (k \neq m) \tag{25}$$

$\because$ $\displaystyle\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_m} = h_k^2 \delta_{km}$ ($\because$ Eq. (21))

$(k = m)$

$$\frac{\partial}{\partial q_k}\left(\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k} h_k^2$$

① $$\cancel{2}\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial q_k} = \cancel{2} h_k \frac{\partial h_k}{\partial q_k}$$

$$\frac{\partial}{\partial q_m}\left(\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_k}\right) = \frac{\partial}{\partial q_m} h_k^2$$

② $$\cancel{2}\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial}{\partial q_m}\frac{\partial x_i}{\partial q_k} = \cancel{2} h_k \frac{\partial h_k}{\partial q_m}$$

$(k \neq m)$

$$\frac{\partial}{\partial q_k}\left(\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial x_i}{\partial q_m}\right) = 0$$

$$\sum_i \frac{\partial^2 x_i}{\partial q_k^2}\frac{\partial x_i}{\partial q_m} + \sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial^2 x_i}{\partial q_k \partial q_m} = 0$$

③ $$\therefore \sum_i \frac{\partial x_i}{\partial q_k^2}\frac{\partial x_i}{\partial q_m} = -\sum_i \frac{\partial x_i}{\partial q_k}\frac{\partial^2 x_i}{\partial q_k \partial q_m}$$

$//$

(Lemma) Contracted derivatives of directional vectors

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} = -\frac{1}{h_k^3} \frac{\partial h_k}{\partial \xi_k} \tag{26}$$

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_m} \frac{\partial \xi_k}{\partial x_i} = -\frac{1}{h_k^3} \frac{\partial h_k}{\partial \xi_m} \tag{27}$$

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_m}{\partial x_i} = -\sum_i \frac{\partial \xi_m}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} = \frac{1}{h_k h_m^2} \frac{\partial h_k}{\partial \xi_m} \tag{28}$$

$\because$ $$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_m}{\partial x_i} = \frac{1}{h_k^2} \delta_{km} \qquad (\because \text{Eq. (21)})$$

$(k = m)$

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_k}{\partial x_i} = \frac{1}{h_k^2}$$

$\frac{\partial}{\partial \xi_k} \times$

① $$\cancel{2} \sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} = -\frac{\cancel{2}}{h_k^3} \frac{\partial h_k}{\partial \xi_k}$$

$\frac{\partial}{\partial \xi_m} \times$

② $$\cancel{2} \sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_m} \frac{\partial \xi_k}{\partial x_i} = -\frac{\cancel{2}}{h_k^3} \frac{\partial h_k}{\partial \xi_m}$$

$(k \neq m)$

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial \xi_m}{\partial x_i} = 0$$

$\frac{\partial}{\partial \xi_k} \times$

$$\sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_m}{\partial x_i} + \sum_i \left(\frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} \right) \frac{\partial \xi_m}{\partial x_i} = 0$$

③ $$\therefore \sum_i \frac{\partial \xi_k}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_m}{\partial x_i} = -\sum_i \frac{\partial \xi_m}{\partial x_i} \frac{\partial}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i}$$

Strategy

$$\partial (D D^{-1}) = 0$$

$$(\partial D) D^{-1} + D \partial D^{-1} = 0$$

$$\therefore \partial D^{-1} = -D^{-1} (\partial D) D^{-1}$$

Note

$$\frac{\partial}{\partial q_k}\sum_i \frac{\partial q_k}{\partial x_i}\frac{\partial x_i}{\partial q_\ell} = \frac{\partial}{\partial q_k}\delta_{k\ell} = 0$$

$$\sum_i \left(\frac{\partial}{\partial q_k}\frac{\partial q_k}{\partial x_i}\right)\frac{\partial x_i}{\partial q_\ell} + \sum_i \frac{\partial q_k}{\partial x_i}\frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial q_\ell} = 0$$

$$\sum_{i\ell}\left(\frac{\partial}{\partial q_k}\frac{\partial q_k}{\partial x_i}\right)\underbrace{\left(\frac{\partial x_i}{\partial q_\ell}\frac{\partial q_\ell}{\partial x_j}\right)}_{\delta_{ij}} = -\sum_{i\ell}\frac{\partial q_k}{\partial x_i}\left(\frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial q_\ell}\right)\frac{\partial q_\ell}{\partial x_j}$$

$$\therefore \quad \frac{\partial}{\partial q_k}\frac{\partial q_k}{\partial x_j} = -\sum_{i\ell}\frac{\partial q_k}{\partial x_i}\left(\frac{\partial}{\partial q_k}\frac{\partial x_i}{\partial q_\ell}\right)\frac{\partial q_\ell}{\partial x_j}$$

or

$$\frac{\partial}{\partial q_k}\frac{\partial q_k}{\partial x_i} = -\sum_{j\ell}\frac{\partial q_k}{\partial x_j}\left(\frac{\partial}{\partial q_k}\frac{\partial x_j}{\partial q_\ell}\right)\frac{\partial q_\ell}{\partial x_i}$$

$\sum_i \frac{\partial q_m}{\partial x_i}\times$

$$\sum_i \frac{\partial q_m}{\partial x_i}\frac{\partial}{\partial q_k}\frac{\partial q_k}{\partial x_i} = -\sum_{ij\ell}\frac{\partial q_k}{\partial x_j}\left(\frac{\partial}{\partial q_k}\frac{\partial x_j}{\partial q_\ell}\right)\overbrace{\frac{\partial q_\ell}{\partial x_i}\frac{\partial q_m}{\partial x_i}}^{\frac{1}{h_m^2}\delta_{m\ell}}$$

$$= -\sum_j \frac{\partial q_k}{\partial x_j}\left(\frac{\partial}{\partial q_k}\frac{\partial x_j}{\partial q_m}\right)\cdot\frac{1}{h_m^2}$$

$$\frac{1}{h_k}e_j^{(k)} = \frac{1}{h_k^2}\frac{\partial x_j}{\partial q_k}$$

$$= -\frac{1}{h_k^{\cancel{2}}h_m^2}\underbrace{\sum_j \frac{\partial x_j}{\partial q_k}\left(\frac{\partial}{\partial q_k}\frac{\partial x_j}{\partial q_m}\right)}_{\cancel{h_k}\frac{\partial h_k}{\partial q_m}} \quad (\because \text{Eq.}(25))$$

$$= -\frac{1}{h_k h_m^2}\frac{\partial h_k}{\partial q_m} \quad //$$

– Gradient Represented along the Local Axis

$$\boxed{\nabla\psi = \sum_{k=1}^{3} \mathbf{e}^{(k)} \frac{\partial}{h_k\partial\xi_k}\psi} \qquad (29)$$

$\because$ $$\frac{\partial}{\partial x_i}\psi = \sum_k \underbrace{\frac{\partial\xi_k}{\partial x_i}}_{\frac{1}{h_k}e_i^{(k)}} \frac{\partial}{\partial\xi_k}\psi \qquad //$$

– Divergence in Terms of the Local Axis

$$\nabla\cdot\mathbf{V} = \sum_i \frac{\partial}{\partial x_i} V_i = \frac{1}{J}\sum_k \frac{\partial}{\partial\xi_k}\left(\frac{J}{h_k}V^{(k)}\right) \qquad (30)$$

where

$$V^{(k)} = \mathbf{e}^{(k)}\cdot\mathbf{V} = \sum_i e_i^{(k)} V_i \qquad (31)$$

$\because$ $\sum_i \frac{\partial}{\partial x_i} V_i$

$$= \sum_{ik} \frac{\partial\xi_k}{\partial x_i}\frac{\partial}{\partial\xi_k}\sum_m e_i^{(m)} V^{(m)}$$

$$= \sum_{ikm} \frac{\partial\xi_k}{\partial x_i}\frac{\partial}{\partial\xi_k}\left[\frac{1}{h_m}\frac{\partial x_i}{\partial\xi_m}V^{(m)}\right]$$

$$\frac{1}{h_k}e_i^{(k)} = \frac{1}{h_k^2}\frac{\partial x_i}{\partial\xi_k}$$

$$= \sum_{km}\frac{\partial}{\partial\xi_k}\left[\frac{1}{h_m}V^{(m)}\right]\underbrace{\sum_i \frac{\partial\xi_k}{\partial x_i}\frac{\partial x_i}{\partial\xi_m}}_{\delta_{km}} + \sum_{km}\frac{1}{h_m}V^{(m)}\sum_i \frac{\partial\xi_k}{\partial x_i}\frac{\partial}{\partial\xi_k}\frac{\partial x_i}{\partial\xi_m}$$

$$= \sum_k \frac{\partial}{\partial\xi_k}\left[\frac{1}{h_k}V^{(k)}\right] + \underbrace{\sum_{km}\frac{1}{h_k^2 h_m}V^{(m)}\underbrace{\sum_i \frac{\partial x_i}{\partial\xi_k}\frac{\partial}{\partial\xi_k}\frac{\partial x_i}{\partial\xi_m}}_{h_k\frac{\partial h_k}{\partial\xi_k}\delta_{km} + h_k\frac{\partial h_k}{\partial\xi_m}(1-\delta_{km}) = h_k\frac{\partial h_k}{\partial\xi_m}}}_{\sum_m \frac{1}{h_m}V^{(m)}\underbrace{\sum_k \frac{1}{h_k}\frac{\partial h_k}{\partial\xi_m}}_{\frac{1}{J}\frac{\partial J}{\partial\xi_m}}}$$

($\because$ Eqs.(24)–(25))

($\because$ Eq.(23))

$$= \sum_k \frac{\partial}{\partial\xi_k}\left[\frac{1}{h_k}V^{(k)}\right] + \sum_k\left[\frac{1}{h_k}V^{(k)}\right]\frac{1}{J}\frac{\partial J}{\partial\xi_k} = \frac{1}{J}\sum_k\frac{\partial}{\partial\xi_k}\left[J\cdot\frac{V^{(k)}}{h_k}\right] \qquad //$$

– Laplacian

$$\nabla^2\psi = \nabla\cdot(\nabla\psi)$$

$$= \frac{1}{J}\sum_k \frac{\partial}{\partial q_k}\Big[\frac{J}{h_k}\underbrace{(\nabla\psi)^{(k)}}_{\frac{1}{h_k}\frac{\partial}{\partial q_k}\psi}\Big]$$

$$\boxed{\therefore\ \nabla^2\psi = \frac{1}{J}\sum_k \frac{\partial}{\partial q_k}\left(\frac{J}{h_k^2}\frac{\partial}{\partial q_k}\psi\right)} \qquad (32)$$

In spherical coordinates, $J = 1\cdot r\cdot r\sin\theta = r^2\sin\theta$ so that

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)\right]$$

$$\boxed{\therefore\ \nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]} \qquad (33)$$